A Model of Systemic Bank Runs*

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Abstract

We develop a tractable model of systemic bank runs. The market-based banking system features a two-layer structure: banks with heterogeneous fundamentals face potential runs by their creditors while they trade marketable assets in the asset (interbank) market in response to creditor withdrawals. The possibility of a run on a particular bank depends on its assets' interim liquidation value, and this value depends endogenously in turn on the status of other banks in the asset market. The within-bank coordination problem among creditors and the cross-bank price externality feed into each other. A small shock can be amplified into a systemic crisis.

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At the heart of the financial crisis of 2007 to 2009 was a series of bank runs that caused failures or impairments of many financial institutions (Bernanke (2010), Gorton (2010), Gertler, Kiyotaki, and Prestipino (2020)).[1] Empirical research has revealed two key facts about how the crisis evolved.

FACT 1 (*Co-movement between bank runs and asset prices*): Since summer 2007, runs on financial institutions had been on a steady rise and, at the same time, "discount rates" in asset (credit) markets — fire-sale discounts on asset-backed securities, repo rates for securitized bonds, and interbank lending rates — exhibited an upward trend.[2] Gorton and Metrick (2012) and Covitz, Liang, and Suarez (2013) document that the probability of runs at the bank or program level was strongly correlated with the LIBOR-OIS spread, a primary measure of interbank lending rates.

FACT 2 (*Sharp crash*): The initial gradual deterioration was followed by a sharp crash — a jump discontinuity in asset prices together with systemic bank runs in September 2008, in the absence of any apparent large exogenous shock to economic fundamentals (see also Gertler, Kiyotaki, and Prestipino (2020)). In an immediate and aggressive response, the Federal Reserve implemented a series of unconventional interventions to boost asset market liquidity (Bernanke (2009)).

Why do bank runs co-move with asset prices? And what gives rise to a sharp crash? Despite a flourishing literature on banking and liquidity crises, few theoretical papers analyze the joint phenomena and offer explanations for the nonlinear events. Indeed, theory on bank runs pioneered by Bryant (1980) and Diamond and Dybvig (1983) typically does not feature a link to asset prices. In recent decades, however, the banking system has experienced a transition toward a "market-based" banking model, in which banks rely greatly on short-term runnable nonretail funding in the capital markets as a source of financing and they heavily trade marketable assets (directly or through repo contracts) in asset and credit markets in response to creditor withdrawals (see, e.g., Brunnermeier (2009) and Gorton (2010)). Financial institutions such as commercial and investment banks, broker dealers, and shadow banks often fish liquidity from the same pool (Shin (2009)).

In this paper we develop a tractable framework that links bank runs with asset prices and demonstrates the amplification mechanisms to explain a systemic crisis. Specifically, we model bank runs in a market-based banking system, in which there are many banks and these banks, with marketable assets and runnable debt, share a common asset market (e.g., asset liquidation market, repo market, and interbank lending market). The model shows how a run on one bank affects, and is affected by, runs on other banks via the asset market, and how a small shock can be amplified into a systemic crisis featuring widespread runs concomitant with collapses in asset prices. The

[1] Besides well-documented runs on commercial banks and investment banks, the modern-day bank runs occurred in the shadow banking system, such as the repo market (Copeland, Martin, and Walker (2014), Gorton and Metrick (2010a, 2010b, 2012), Krishnamurthy, Nagel, and Orlov (2014)), money market mutual funds (Duygan-Bump et al. (2013)), and the ABCP market (Covitz, Liang, and Suarez (2013), Kacperczyk and Schnabl (2010), Acharya and Schnabl (2010)).

[2] All these rates are "discount rates" used in pricing assets.

model explains the empirical facts and has novel policy implications.

We begin by presenting a baseline three-date model. There is a continuum of financial institutions ("banks"). At the initial date, each bank finances its long-term risky asset with short-term debt from many creditors. At the interim date, each bank realizes its asset's fundamental value (i.e., asset quality) — the expected payoff at the final date. A bank's creditors receive noisy private signals about the bank's fundamental value and decide whether to roll over or to withdraw. If too many creditors of a bank withdraw, the bank will be unable to satisfy these early withdrawals and hence will fail at the interim date. All failing banks liquidate their assets in a common asset market. The liquidation value of a bank therefore depends on its asset's fundamental value, on the aggregate volume of liquidation (fire sales) in the system, and on market liquidity (depth). Under this setting, the equilibrium is characterized by the joint determination of three endogenous variables: the rollover decision of creditors, the interim liquidation value of a bank in the case of its being run, and the aggregate liquidation in the system. In particular, when a bank's creditors make rollover decisions, they need to form expectations about the liquidation value of the bank, because the value determines the extent to which the bank can withstand early withdrawals (i.e., the degree of fragility to runs) and in turn the level of coordination risk among peer creditors.

The interplay between the within-bank coordination game and the cross-bank price externality gives rise to two-way feedback between liquidation values and creditor runs. If creditors of a bank believe the liquidation value of the bank's asset to be low, they will optimally choose to run more often because a lower liquidation value increases within-bank coordination risk. In turn, if creditors run more often, more banks will fail in the system, increasing aggregate fire sales and thus reducing every bank's liquidation value due to the fire-sale externality. The feedback effect is also strong enough to potentially generate multiple equilibria. In fact, because banks share a common asset liquidation market, strategic complementarities arise among creditors of different banks, in addition to strategic complementarities among creditors of the same bank. The increased degree of complementarity among creditors in the system results in a higher likelihood of equilibrium multiplicity. The amplification mechanism above — the feedback loop as well as the possibility of multiplicity jumps — implies that a small shock to the aggregate state variables (i.e., the average asset quality of banks in the system and the asset market depth) can turn into a systemic crisis that exhibits a large number of bank runs accompanied by a sharp fall in asset prices.

We next present the full model. The full model allows banks to hold liquid assets in addition to their holding of illiquid assets. The presence of liquid assets on the balance sheet of banks not only directly affects the run incentives of bank creditors but also changes the behavior of the asset market. Banks that realize stronger fundamentals and thus face fewer interim withdrawals in equilibrium will supply liquidity to the asset market, while banks that realize weaker fundamentals will demand liquidity. That is, interbank trading arises endogenously. The liquidation of illiquid assets of banks in the asset market features both fire sales to outside investors and interbank trading; in other words, the two types of trading co-exist. In particular, creditor runs, fire-sale prices, and interbank rates are jointly determined in equilibrium. The full model hence sheds light on the empirical facts by additionally rationalizing the movement of interbank (repo) rates, as in the data. Equally importantly, the full model with liquid asset holdings, including heterogeneous holdings across banks, lays the groundwork for analysis of realistic policy interventions.

The full model also allows for aggregate uncertainty. In the baseline model with no aggregate uncertainty, the average asset quality of banks in the system — the aggregate state — is known, so creditors perfectly foresee the aggregate liquidation under rational expectations. In contrast, in the full model, the aggregate state is stochastic. In this case, the liquidation value of a bank, as a function of the aggregate liquidation or the aggregate state, is stochastic. Individual creditors use their private signals about their bank (i.e., local information) to infer the fundamental value of their bank as well as the aggregate or global state. In other words, individual creditors' private signals reveal information about the status of not only their own bank but also other banks. Since one signal plays dual roles to infer two-layer, correlated fundamentals, solving the global-games model is challenging; nevertheless, we solve the model analytically.

The full model with aggregate uncertainty delivers additional new insights. An increase in uncertainty about aggregate fundamentals (akin to "fear," "panic," etc.) — the second moment, other than a change in fundamentals per se — the first moment, can play an important role in amplification mechanisms responsible for a systemic crisis. First, aggregate uncertainty amplifies the within-bank coordination problem, precipitating individual creditors to run. In fact, in the system context, a bank essentially faces a run by its own creditors as well as "runs" by other banks via the asset market. Uncertainty about cross-bank "runs" aggravates the within-bank coordination problem for every bank, triggering actual cross-bank "runs" with an amplification loop. Such an amplification mechanism would not exist in a single-bank model without an asset market connecting many banks. Second, an increase in aggregate uncertainty is typically accompanied by an increase in bank-level fundamental dispersion, as empirical literature documents. If cross-bank fundamental dispersion increases more than does aggregate uncertainty, the equilibrium can switch from uniqueness to multiplicity, and thus a self-fulling multiplicity jump is possible. In fact, when cross-bank fundamental dispersion increases, private signals about one bank will become less informative about other banks, so creditors will rely less on their private signals about their own bank and more on the prior (public signal) to infer the aggregate state, in which case equilibrium multiplicity becomes more likely.

Our model offers new and unique policy implications. Concerning ex post policies, our framework, explicitly modelling bank runs jointly with asset markets, allows us to study and evaluate two major unconventional intervention measures that the Federal Reserve implemented: providing liquidity support to asset markets and injecting liquidity into financial institutions. We analyze the pros and cons of each measure, demonstrate the tradeoff, and show the optimal combination. Our analysis highlights that liquidity support to the asset market is effectively a price-contingent bailout. This measure is particularly effective in tackling expectations-driven systemic runs: when a bad expectation of creditors precipitates their running, driving down asset prices, it is precisely at that time that the government's liquidity support, for a given aggregate amount, is able to buy more assets, which thereby provides a cushion making asset prices not fall as much, so a bad expectation may not be formed in the first place since it would not be justified. Overall, our model suggests that for a crisis of high severity it is optimal for the government to use its limited resources to support the asset market only, for a crisis of low severity supporting banks is optimal, and for a crisis of medium severity it is optimal to mix the two measures. As for ex ante policies, our model studies the ex ante problem, endogenizes liquid asset holdings of banks at the initial date, and addresses the question of whether individual banks' decision is socially optimal. This provides novel perspectives on some pillar macroprudential tools underscored in Basel III.

Related literature. The literature on bank runs is vast. Our paper contributes to this literature by modelling systemic bank runs — runs that occur simultaneously on many banks which interact in a common asset market. Our focus is on studying the interplay between the within-bank coordination among creditors and the cross-bank interaction in the asset market, which is crucial to explain the empirical facts and shed light on policy measures. In contrast, existing papers typically model one element only. For example, Diamond and Dybvig (1983) and Gertler, Kiyotaki, and Prestipino (2020) treat the entire banking system as a single or representative bank without considering the interactions among banks and take asset prices largely as exogenous,[3] while papers such as Uhlig (2010), Bernardo and Welch (2004), and Morris and Shin (2004) directly study market-wide runs and abstract away the within-bank coordination problem by assuming that each bank has one representative creditor or simply that each firm/bank is an individual with zero debt.[4]

Our paper contributes to our understanding of amplification mechanisms in banking and liquidity crises. Brunnermeier (2009), Krishnamurthy (2010), and Brunnermeier and Oehmke (2013) survey various amplification mechanisms, where one prominent mechanism highlighted is the feedback between asset prices and balance sheet constraints: lower asset prices tighten balance sheet constraints, causing asset liquidations, further depressing asset prices. Our paper studies and formalizes the amplification mechanism between systemic bank runs and asset prices under stress. We show that a feedback loop exists between asset prices and creditors' run on individual banks

[3] The liquidation value of bank assets in Diamond and Dybvig (1983) is exogenous. Interbank trading and prices are absent in Gertler, Kiyotaki, and Prestipino (2020).

[4] Diamond and Rajan (2005) study banking crises and aggregate liquidity shortages, where there are no interbank market or coordination issues among creditors within a bank. Allen and Gale (2000) study financial contagion through interbank contracting claims and assume that banks face exogenous idiosyncratic liquidity shocks, where there is no liquidity trading of banks in a common asset market or endogenous illiquidity risk for a bank.

(i.e., within-bank coordination). Unlike existing models in which there are often explicit exogenous price-dependent constraints (e.g., the collateral constraint in Kiyotaki and Moore (1997) and the VaR constraint in Brunnermeier and Pedersen (2009)), in our model there are no explicit constraints and instead asset prices endogenously affect the *coordination risk* and run incentives of bank creditors. The key state variable that triggers the amplification loop in our model is asset fundamentals (akin to the net worth in Bernanke and Gertler (1989) and Kiyotaki and Moore (1997)), or the risk-absorbing capacity of asset market investors as in He and Krishnamurthy (2012).

Our paper is related to the literature that studies bank runs using global-games methods. The models in this literature realistically assume that creditors receive private signals about a bank's fundamentals, and show that a bank run is both fundamental-based and panic-based.[5] Early works in this literature study runs on a single bank and take the interim liquidation value (function) of the bank as exogenous (e.g., Rochet and Vives (2004), Goldstein and Pauzner (2005), and the general model of Vives (2014a)). Liu (2016) and Eisenbach (2017) recently made progress in endogenizing the interim liquidation values by studying many banks with interactions in a common asset market.[6] Our paper contributes to this literature in i) building a tractable framework to endogenize liquidation values in the financial system context, featuring both fire sales to outside investors and interbank trading, which is important for explaining empirical evidence and policy analysis, and ii) modelling systemic bank runs in a general setting without and with aggregate uncertainty. When aggregate uncertainty is present, our model studies the realistic setting in which individual creditors use their private signals about their bank to infer both the fundamentals of their bank and the aggregate state of the banking system (considering that local information is often more available and precise than global information). Studying this kind of game and analytically solving the equilibrium is new to the literature and constitutes a methodological contribution.

A new paper by Goldstein et al. (2020) presents a global-games model to study bank heterogeneity and financial stability by emphasizing reinforcement of two complementarities. Their model also uses a setting in which runs on banks are connected because of fire sales in a common asset market, which is closely related to Liu (2016, 2018a,b), Eisenbach (2017), and our work. Their paper considers aggregate uncertainty and uses an information structure to have a unique equilibrium. Specifically, their model assumes that individual creditors receive *two* distinct private signals: a virtually perfect signal about the bank-specific component of their bank's asset fundamentals and a noisy signal about the aggregate component. Under this information structure, creditors of all banks face one uncertainty, which is about the aggregate state, and there exists a unique equilibrium (see also Sákovics and Steiner (2012)). In contrast, our model assumes that individual

[5] See, e.g., Gorton and Winton (2003), Allen and Gale (2009), and Goldstein (2013) for discussions of evidence.

[6] Eisenbach (2017) uses a reduced-form approach by assuming that outside investors provide an exogenous downward-sloping aggregate demand curve for bank assets (while his paper has other focuses). Liu (2016) explicitly models interbank trading.

creditors receive only one private signal, which is about their bank's overall asset fundamentals (instead of each component), following the standard literature. Individual creditors thus face two uncertainties and use one signal to infer two layers of fundamentals — their bank's fundamentals and the aggregate state, and both a unique equilibrium and multiple equilibria are possible.

Prior research such as Hellwig (2002) shows equilibrium multiplicity in coordination games if the precision of the public signal increases sufficiently faster than the precision of the private signals. In our model, the global game is embedded into a Walrasian economy, in which case multiple equilibria can emerge even when the precision of private signals approaches the limit of infinity. Ozdenoren, Yuan, and Zhang (2018) and Asriyan, Fuchs, and Green (2019) show that the intertemporal coordination in the infinite horizon can give rise to multiple self-fulfilling stationary equilibria. In their models there is feedback between a higher (lower) asset price and a pooling (separating) equilibrium. Multiplicity in our model does not depend on intertemporal coordination or complementarity between participation and leverage (Gârleanu, Panageas, and Yu (2015)). Goldstein, Ozdenoren, and Yuan (2013) and Sockin and Xiong (2015), among others, show the informational feedback of financial market prices. Asset prices in our model instead affect coordination (illiquidity) risk.

Our paper is also related to several studies on dynamic runs. He and Xiong (2012) study dynamic debt runs in a model in which a firm has a time-varying fundamental and a staggered debt structure. They consider one firm, not various firms (banks), and every creditor observes the same public information. Their work formalizes the intertemporal coordination problem among creditors of a firm.[7] Our paper focuses instead on studying the interaction of many banks with endogenous liquidation values. He and Li (2021) develop a dynamic model in which an entrepreneur borrows from OLG households via layers of funds with rollover (run) problems between layers akin to He and Xiong (2012). Our model features a different two-layer structure, namely, asset market — banks — creditors. Martin, Skeie, and von Thadden (2014a, 2014b) study runs on short-term funding markets. Their paper (2014a) shows how market fragility due to sudden collective expectation changes can be addressed by individual banks and regulators. Their paper (2014b) focuses on how different market microstructures influence expectation-driven runs. Our model studies systemic bank runs and runs are both fundamental-based and expectation (panic)-based.

Our paper is additionally connected to the literature on pecuniary externalities and fire sales (e.g., Gromb and Vayanos (2002), Caballero and Krishnamurthy (2003), Lorenzoni (2008), He and Kondor (2016), Dávila and Korinek (2018), and the surveys of Krishnamurthy (2010) and Brunnermeier, Eisenbach, and Sannikov (2013)). In these models, individual agents' actions can

[7]Angeletos, Hellwig, and Pavan (2007) study dynamic global games of regime change. Choi (2014) simplifies the rollover decision within a bank and does not model the run in the regime-switching game. In studying trading dynamics with search, Chiu and Koeppl (2016) show a government's role in improving coordination, consistent with our finding.

cause constraints to tighten for other agents and hence trigger asset transfers in the economy that are often inefficient. Our model shares some similarities with these models, but there are no explicit price-dependent constraints in our model. In this sense, fire sales in our model are endogenous.

The paper is organized as follows. Section I presents the baseline model. Sections II and III study the full model. Section IV analyzes policy implications. Section V concludes.

I. Baseline Model

In this section, we present the baseline model, to highlight the model framework.

A. Setting

Figure 1 illustrates the framework; the notation will be explained in due course. There are three dates: $t = 0$, 1, and 2. We discuss banks, the asset market, and creditor runs, in order.

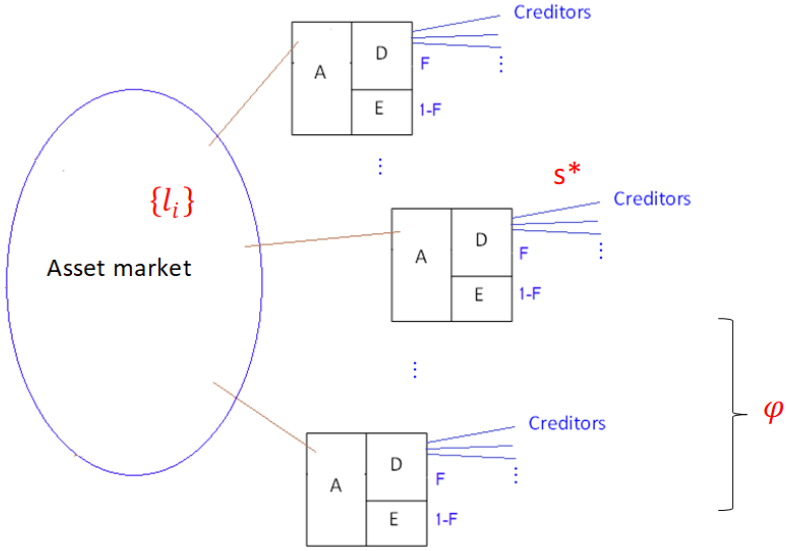


Figure 1. The model framework.

A.1. Banks

There is a continuum of banks with unit mass, indexed by $i \in [0, 1]$. At $t = 0$, each bank invests in one unit of its own assets at a cost of 1. The cost is financed from two sources: an amount F comes from a continuum of its creditors (depositors) with mass F, with each creditor contributing 1, and an amount $1 - F$ comes from its equityholder (bankowner).[8] At $t = 1$, a creditor of a bank has the right to decide whether or not to roll over lending to the bank. If he decides not to roll over, his claim is the par value 1 at $t = 1$; if, instead, he decides to roll over, the (promised) notional

[8]We assume that each bank has its own creditor base (for example, regional banks).

claim to him is R at $t = 2$, where $R > 1$ is the gross interest rate.[9] Bankowners and bank creditors are risk-neutral.

The payoff of bank i's assets at $t = 2$ is $v_i = \theta_i + e_i$, which follows a normal distribution as in Grossman and Stiglitz (1980). That is, the uncertainty of the payoff is resolved gradually. Specifically, the term θ_i, interpreted as asset quality, has its realization at $t = 1$ with $\theta_i = \mu_\theta + \sigma_\theta \delta_i$ (denoting $\tau_\theta \equiv 1/\sigma_\theta^2$), where μ_θ corresponds to the aggregate state of the economy at $t = 1$ and δ_i, which is independently drawn from the identical distribution $\delta_i \sim N(0, 1)$ across $i's$, corresponds to the bank-specific realization. In the baseline model of this section, we assume that μ_θ is a constant and is common knowledge, that is, there is no aggregate uncertainty. The term e_i is a random variable with distribution $e_i \sim N(0, \sigma_e^2 = \tau_e^{-1})$ and its uncertainty is resolved at $t = 2$. For simplicity, we assume that $e_i \equiv e$ is perfectly correlated across banks.[10] Figure 2 illustrates the timeline of the asset payoff's uncertainty resolution.

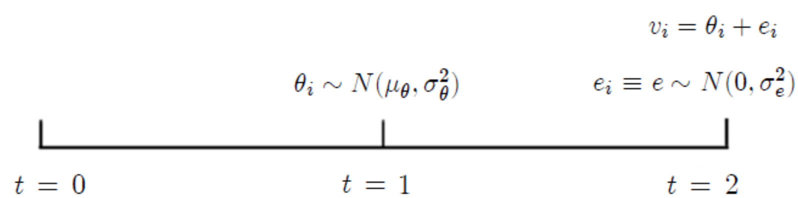


Figure 2. Timeline of the asset payoff's uncertainty resolution.

Although the asset quality of a bank is realized at $t = 1$, its creditors are not informed. Nevertheless, at $t = 1$ creditors of a bank receive imperfect information (signals) about the asset quality of the bank. Specifically, the signal for creditor h of bank i (about asset quality θ_i) at $t = 1$ is $s_i^h = \theta_i + \sigma_s \epsilon_i^h$, where $\sigma_s > 0$ is a constant (denoting $\tau_s \equiv 1/\sigma_s^2$), the creditor-specific noise $\epsilon_i^h \sim N(0, 1)$, ϵ_i^h is independent across h's for a given i, and each ϵ_i^h for a given i is independent of θ_i.

A.2. Asset Market

If a bank suffers a creditor run (to be elaborated), its assets must be liquidated or put on fire sales at $t = 1$ in a competitive asset market, which consists of a continuum of competitive investors with a total mass of n. Investor $j \in [0, n]$ has utility function $U(W^j) = -\exp\left(-\gamma W^j\right)$, where W^j is end-of-period wealth at $t = 2$ and γ is the risk-aversion (CARA) coefficient. Without loss of generality, the gross risk-free interest rate between $t = 1$ and 2 is normalized to 1.

Investors have private information (signals) about banks' asset qualities. Specifically, the signal for investor j about asset quality θ_i at $t = 1$ is $x_i^j = \theta_i + \sigma_x \varepsilon_i^j$, where $\sigma_x \geq 0$ is a constant, the

[9] Without loss of generality, we normalize the interim notional claim to 1. What matters for the model is the interest rate between $t = 1$ and $t = 2$, that is, R.

[10] As long as e_i is correlated across banks to some degree (i.e., not perfectly diversified away, for instance, under the assumption that there exists a common risk factor across banks' assets), our model result of a downward-sloping liquidation price (Lemma 1) and in turn other results will hold.

investor-specific noise $\varepsilon_i^j \sim N(0,1)$, ε_i^j is independent across i's and j's, and each ε_i^j for a given i is independent of θ_i.

Suppose that in equilibrium banks with a total mass of φ ($\in [0,1]$) suffer creditor runs in the system. The total measure of bank assets in the system under liquidation is then φ.[11] Denote by l_i the liquidation price of bank i's assets at $t = 1$.

A.3. Creditor Runs

Consider a typical bank i. If greater than $\frac{l_i}{F}$ proportion of its creditors decline to roll over their lending at $t = 1$, the bank's liquidation value will not be sufficient to cover these creditors' claims, leading to its failure (we call this scenario a "creditor run"). Alternatively, one may think of l_i as the collateral value of the bank's assets. This means that the bank can raise cash of at most l_i at $t = 1$ by pledging its assets as collateral. If the demand for cash exceeds l_i at $t = 1$, the bank will fail.

A creditor's payoff depends crucially on the actions of other creditors of the same bank. Let λ denote the proportion of creditors of a bank who choose not to roll over (i.e., choose to call loans). The payoff for a particular creditor is then given as in Figure 3.

	Total calling proportion $\lambda \in [0, \frac{l_i}{F})$ (bank survives)	Total calling proportion $\lambda \in [\frac{l_i}{F}, 1]$ (bank fails)
Hold	$\min\left[R, \frac{v_i}{F}\right]$	$\frac{l_i}{F} - \Delta$
Call	1	$\frac{l_i}{F}$

Figure 3. Creditor-run payoff structure.

If $\lambda \in [\frac{l_i}{F}, 1]$, a creditor run occurs and the bank fails at $t = 1$. In this case, *all* creditors equally share the liquidation value l_i at $t = 1$, but those who have not called will incur fee Δ (e.g., legal cost, agency cost, or reputation loss) to get their money back. This setup of a first-mover advantage of withdrawing (calling) follows that in Eisenbach (2017). As Eisenbach argues, the first-mover advantage, not depending on the sequential service constraint inherent in deposit contracts, is more representative of market-based funding without a sequential service constraint as in Cole and Kehoe (2000).[12] Moreover, as will become clear later, under the limit $\sigma_s \to 0$ all creditors of a bank are in the same position in equilibrium, i.e., either all of them run on the bank or none of

[11] We will focus on the case $\sigma_s \to 0$. In equilibrium, then, if a bank suffers a creditor run, it will liquidate its assets *entirely* (i.e., no partial liquidation).

[12] This bank-run payoff structure is in the same spirit as that in Rochet and Vives (2004). In Appendix B, we show the robustness of our model under the alternative payoff structure of Rochet and Vives (2004).

them does so ($\lambda = 1$ or 0). Therefore, paying the extra fee Δ under the scenario of "bank failing, holding" is *off* the equilibrium path, which makes a general equilibrium analysis convenient.

If $\lambda \in [0, \frac{l_i}{F})$, we follow Morris and Shin (2009) to simplify the payoff structure. Concretely, if a bank has less than $\frac{l_i}{F}$ proportion of its creditors calling, partial liquidation will occur but the bank can still survive to $t = 2$, in which case Morris and Shin (2009) assume that the bank's balance sheet reverts to its initial state. Essentially, after creditor withdrawals that do not result in bank failure, the asset side of the bank's balance sheet is restored to v_i and the liability side reverts to the total notional debt value FR claimed by F creditors. Put differently, as long as $\lambda \in [0, \frac{l_i}{F})$, the bank continues as if it had not experienced any withdrawals. We use this simplified payoff structure given in Figure 3 in the main text. We will show in Appendix B that our model results are robust under the full payoff structure as in Diamond and Dybvig (1983).

A.4. Timeline

At $t = 0$, the liability side of a bank's balance sheet is given by $(F,\ 1 - F)$ and the contract term $(1, R)$ with creditors is fixed. At $t = 1$, creditors, with information set $\left\{\mu_\theta, s_i^h\right\}$, move *first* by making their rollover decisions, and banks move *later* by conducting asset sales in the asset market based on the total withdrawals requested by their creditors, at which stage asset prices $\{l_i\}$ are formed. The rollover decisions of bank i's creditors cannot be conditioned on the asset price l_i. First, when creditors make their rollover decisions, the asset prices $\{l_i\}$ will not have been formed yet. Empirically, a bank's creditors, who hold a debt contract, hardly know in advance details of the bank's business operation, such as knowing beforehand the yet-to-be-realized selling price of bank assets. The asset price l_i is also bank-specific, unlike a publicly observable index (which is more like a realized φ in our model). Second, nevertheless, creditors receiving a signal about θ_i is equivalent to receiving a signal about l_i, since l_i is a function of θ_i (as we show in Section I.B.1). In the limiting case of signal precision (i.e., $\sigma_s \to 0$) that we focus on, therefore, creditors of bank i are almost certain about l_i under rational expectations about φ, which will become clear in Section I.B.1. At $t = 2$, payoffs are realized and contracts are delivered. Figure 4 illustrates the timeline.

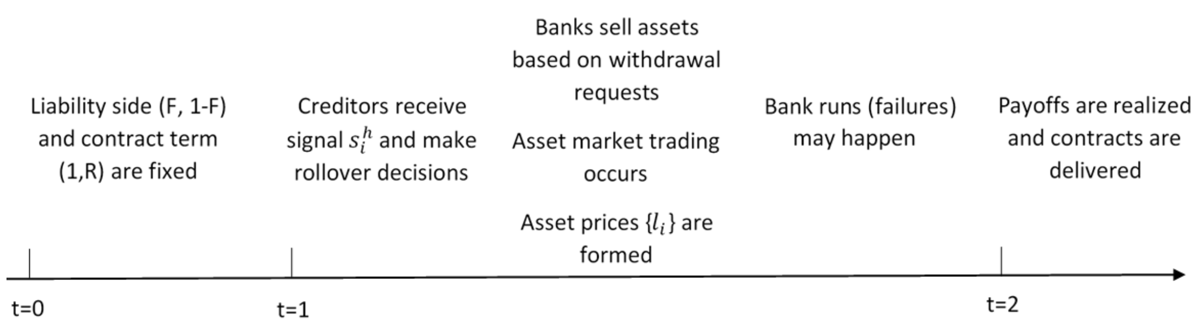


Figure 4. Timeline.

B. Equilibrium

We study the equilibrium at $t = 1$, the point at which creditors make their rollover decisions. We are interested in the equilibrium in which every creditor uses a threshold (monotone) strategy

$$s_i^h \longmapsto \begin{cases} Call & s_i^h < s^* \\ Hold & s_i^h \geq s^* \end{cases},$$

where s_i^h is the signal received by creditor h of bank i and s^* is the rollover threshold. Because banks are identical ex ante, we naturally consider the *symmetric* equilibrium in which creditors of all banks use a common strategy, that is, the threshold s^* is not bank-specific. We will show that an upper dominance region exists for the bank-run game in our model.

At $t = 1$, the portfolio choice of an investor j in the asset market is given by

$$\max_{\{q_i^j\}} \mathbb{E}\left[-\exp\left(-\gamma W^j\right) \mid \left\{x_i^j\right\}, \{l_i\}\right] \qquad \text{s.t.} \quad W^j = \int q_i^j (v_i - l_i) di, \tag{1}$$

where q_i^j is the quantity of investor j's demand for asset i given his information set $\left\{\left\{x_i^j\right\}, \{l_i\}\right\}$.

Definition 1 ***(Creditor run-asset market equilibrium)*** *The creditor run-asset market equilibrium at $t = 1$ is characterized by the triplet $(s^*, \{l_i\}, \varphi)$, where s^* is creditors' rollover threshold, l_i is the liquidation price of bank i's assets, and φ is the total measure of assets under liquidation in the system, such that i) given the price rule l_i and creditors' rational expectations of φ, creditors set their rollover threshold as s^*, ii) given the rollover threshold s^*, the total liquidation is φ, and iii) given the total liquidation φ, the equilibrium liquidation price of bank i's assets is l_i.*

B.1. Solving the Equilibrium

We solve the equilibrium by analyzing its three elements.

Asset market in equilibrium. Financial market can aggregate dispersed information of investors without necessarily resorting to noise traders (Vives (2014b)). We can follow the trading mechanism in Vives (2014b) to obtain the result that the price l_i is a deterministic function of fundamentals θ_i. Equivalently and for simplicity, we focus here on the *fully-revealing equilibrium* of the asset market, i.e., the equilibrium in which financial prices fully reveal the fundamentals of the trading assets in the spirit of Hayek (1945). In our context, it is the equilibrium in which θ_i is fully revealed to investors through the financial price l_i. Alternatively, we can simply assume that the precision of investors' private signals approaches the limit $\sigma_x \to 0$ as in Morris and Shin (2004), just as the precision of creditors' private signals approaches the limit $\sigma_s \to 0$; this gives the same asset market equilibrium as in the case in which investors have perfect information about $\{\theta_i\}$.

Lemma 1 summarizes the asset market equilibrium.

Lemma 1 *Given the aggregate liquidation φ, the liquidation price of bank i's assets is given by*

$$l_i = \theta_i - \varphi/k, \tag{2}$$

where $k \equiv n/\left(\gamma\sigma_e^2\right)$ measures market liquidity (i.e., market depth).

Lemma 1 is in the spirit of Grossman and Miller (1988). When the risk-averse market maker sector is forced to absorb more risky assets, the price of *every* risky asset is affected and reduced because of the limited risk-absorbing capacity of the market maker sector. A fire-sale externality across banks arises as φ is common. The liquidation is socially inefficient because the long-term illiquid assets are prematurely liquidated and transferred from higher-valuation sellers to lower-valuation buyers. As in the large literature, "market liquidity" is measured as *market depth*, k.

Creditor run in equilibrium for an individual bank. Consider a typical bank i. Because l_i is fundamental (θ_i)-dependent by Lemma 1, when θ_i is sufficiently high, bank i will survive even if every one of its creditors withdraws. That is, an upper dominance region exists. Therefore, we only need to focus on threshold equilibria (Morris and Shin (2003) and Vives (2014a)).

Recalling Figure 3, denote by

$$D(\theta_i; R) \equiv \mathbb{E}\left(\min\left[R, \frac{v_i}{F}\right] |\theta_i\right) \tag{3}$$

the expected payoff of the debt at $t = 2$ conditional on the realization of θ_i at $t = 1$, where $v_i \sim N(\theta_i, \sigma_e^2)$. Clearly, we have the properties $\frac{\partial D}{\partial \theta_i} > 0$ and $\lim_{\theta_i \to +\infty} D(\theta_i; R) = R$.

Given that all other creditors of bank i use the threshold s^*, the bank, when realizing asset quality as θ_i, has a $\lambda(\theta_i; s^*) = \Pr(\theta_i + \sigma_s \epsilon_i^h < s^*) = \Phi\left(\frac{s^* - \theta_i}{\sigma_s}\right)$ proportion of its creditors withdrawing, where $\Phi(\cdot)$ stands for the c.d.f. of the standard normal and $\phi(\cdot)$ denotes its p.d.f.. Moreover, the bank with realized asset quality θ_i will have its asset liquidation value to be $l_i = \theta_i - \varphi/k$. Hence, by the nature of creditor runs, the bank's failure threshold, denoted by θ^*, is given by

$$\frac{\theta^* - \varphi/k}{F} = \Phi\left(\frac{s^* - \theta^*}{\sigma_s}\right). \tag{4}$$

That is, the bank fails if and only if $\theta_i < \theta^*$, which is rationally anticipated by individual creditors.

Given the bank's failure threshold θ^*, what is the optimal strategy for an individual creditor h? He rolls over if and only if his signal s_i^h is above the threshold s^*, and, by recalling Figure 3, s^* solves the indifference equation

$$\mathbb{E}_{\theta_i|s_i^h}\left[(D(\theta_i) - 1)\cdot \mathbf{1}_{\theta_i \geq \theta^*} + (-\Delta)\cdot \mathbf{1}_{\theta_i < \theta^*} \middle| s_i^h = s^*\right] = 0, \tag{5}$$

where $\mathbb{E}_{\theta_i|s_i^h}(\cdot|s_i^h)$ is the conditional expectation operator over θ_i, with the conditional distribution being $\theta_i|s_i^h \sim N(\frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta + \frac{\tau_s}{\tau_\theta+\tau_s}s_i^h, \frac{1}{\tau_\theta+\tau_s})$, and $\mathbf{1}_x = \begin{cases} 1 & \text{if } x \text{ is true} \\ 0 & \text{if } x \text{ is false} \end{cases}$ is an indicator function. In equation (5), based on Figure 3, in the case of bank survival ($\theta_i \geq \theta^*$) holding is superior to calling, with the net gain equal to $D(\theta_i) - 1$, and in the case of bank failure ($\theta_i < \theta^*$) holding is inferior to calling, with the net gain equal to $-\Delta$.

We consider the limiting case of signal precision, $\sigma_s \to 0$. Under the limiting case, we prove that the system of equations (4) and (5) is transformed into

$$\theta^* = s^* \tag{6}$$

and

$$(D(s^*) - 1) \cdot \frac{l_i}{F} + (-\Delta) \cdot \left(1 - \frac{l_i}{F}\right) = 0, \tag{7}$$

where $l_i = l_i(\theta_i = s^*) = s^* - \varphi/k$ by (2). The term $\frac{l_i}{F}$ in (7) measures *illiquidity/coordination risk* and the term $D(s^*)$ measures *insolvency risk*. The intuition behind (7) is as follows. In making rollover decisions, an individual creditor faces *fundamental uncertainty* as well as *strategic uncertainty* (i.e., he is not sure how many peer creditors of the same bank will roll over). Under the limit $\sigma_s \to 0$, *fundamental uncertainty* disappears (i.e., $s_i^h \to \theta_i$). Thus, for the marginal creditor who receives signal $s_i^h = s^*$, his inference of θ_i is $\theta_i = s^*$ and his inference of l_i is $l_i(\theta_i = s^*) = s^* - \varphi/k$, given the price rule (2). However, strategic uncertainty does not disappear under $\sigma_s \to 0$. For the marginal creditor, he perceives that λ (i.e., the proportion of peer creditors choosing to call) is uniformly distributed within $[0, 1]$. Hence, in his eyes, the probability that the proportion of creditors calling loans is less than $\frac{l_i}{F}$ is $\frac{l_i}{F}$, that is, in his eyes, the probability of bank survival is $\frac{l_i}{F}$ and that of bank failure is $1 - \frac{l_i}{F}$, by recalling Figure 3.

Rewriting (7) yields

$$\frac{s^* - \varphi/k}{F}\frac{D(s^*) - 1 + \Delta}{\Delta} = 1. \tag{8}$$

The limit $\sigma_s \to 0$ also implies that in equilibrium all creditors of a bank are in the same position ex post — either all of them decide to roll over or none of them does so. This, in turn, implies that in equilibrium a bank either completely liquidates its assets or does not liquidate any fraction, i.e., no partial liquidation.[13] Lemma 2 follows.

Lemma 2 *Given the price rule l_i in (2) and creditors' rational expectations of φ, the creditor-run equilibrium for an individual bank, characterized by (s^*, θ^*), is given by (6) and (7) under $\sigma_s \to 0$. For an exogenous φ, the equilibrium is unique and the comparative statics $\frac{\partial s^*}{\partial \varphi} > 0$ follows.*

It is worth noting that the creditor-run game in our model features a fundamental-dependent

[13] When $\theta_i < s^*$, it follows that $\theta_i - \varphi/k < s^* - \varphi/k < F$ by (8).

liquidation value and a fundamental-dependent payoff structure (i.e., both l_i and $D(\theta_i; R) \equiv \mathbb{E}\left(\min\left[R, \frac{v_i}{F}\right] |\theta_i\right)$ depend on θ_i), and solving the equilibrium is nontrivial. In the literature, Rochet and Vives (2004) use a fundamental-dependent liquidation value but a fundamental-independent payoff structure, while Goldstein and Pauzner (2005) use a fundamental-dependent payoff structure but a fundamental-independent liquidation value. Eisenbach (2017) assumes physical liquidation, where an asset's liquidation value does not depend on its fundamentals (see also Li and Ma (2022)).

Aggregate liquidation in the system. Because the creditors of all banks use the same rollover threshold, the failure threshold is the same for all banks. Recall that the asset quality distribution across banks at $t = 1$ is $\theta_i \sim N(\mu_\theta, \sigma_\theta^2)$. Banks with realized asset quality $\theta_i \geq \theta^*$ will survive at $t = 1$ while all others will fail. Hence, by $\theta^* = s^*$ under $\sigma_s \to 0$, the total measure of failing banks in the system is given by

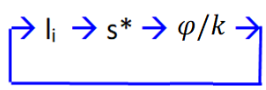
$$\varphi = \Phi\left(\frac{s^* - \mu_\theta}{\sigma_\theta}\right), \tag{9}$$

which also implies that the total measure of bank assets under fire sales is φ under $\sigma_s \to 0$.

Lemma 3 summarizes the results of the three elements above.

Lemma 3 *The creditor run-asset market equilibrium at $t = 1$, characterized by $(s^*, \{l_i\}, \varphi)$ for a given (μ_θ, k), solves the system of equations (2), (7), and (9) under the limit $\sigma_s \to 0$. Two-way feedback exists between liquidation prices (φ and thereby $\{l_i\}$) and the run threshold (s^*): $\frac{\partial s^*}{\partial \varphi} > 0$ in (7) (or equivalently (8)) and $\frac{\partial \varphi}{\partial s^*} > 0$ in (9).*

The two-way feedback in Lemma 3 is intuitive. Essentially, we have the following feedback loop:



When creditors run on banks with a higher threshold, more banks in the system will fail, resulting in a lower liquidation price for every bank. Creditors of a bank have rational expectations on this and thus have higher incentives to run in the first place due to higher *coordination (illiquidity) risk*.

B.2. Characterization of the Equilibrium

We examine the creditor run-asset market equilibrium in Lemma 3, where φ is endogenous. Combining (8) and (9) yields one equation:

$$V \equiv \left\{\frac{1}{F}\left[s^* - \Phi\left(\frac{s^* - \mu_\theta}{\sigma_\theta}\right) \Big/ k\right]\right\} \frac{D(s^*) - 1 + \Delta}{\Delta} = 1. \tag{10}$$

The equilibrium at $t = 1$ is fully characterized by equation (10).[14] Write the left-hand side (LHS)

[14]Based on (7), conceptually, an equilibrium s^* must also satisfy the conditions $0 < \frac{l_i(\theta_i = s^*)}{F} \leq 1$ and $D(s^*) \geq 1$.

of (10) as function $V(s^*;\mu_\theta,k)$. Figure 5 plots equation $V(s^*;\mu_\theta,k)=1$ under a set of parameter values $\sigma_\theta=0.6$, $F=0.6$, $R=1.1$, $\sigma_e=0.2$, and $\Delta=0.1$.

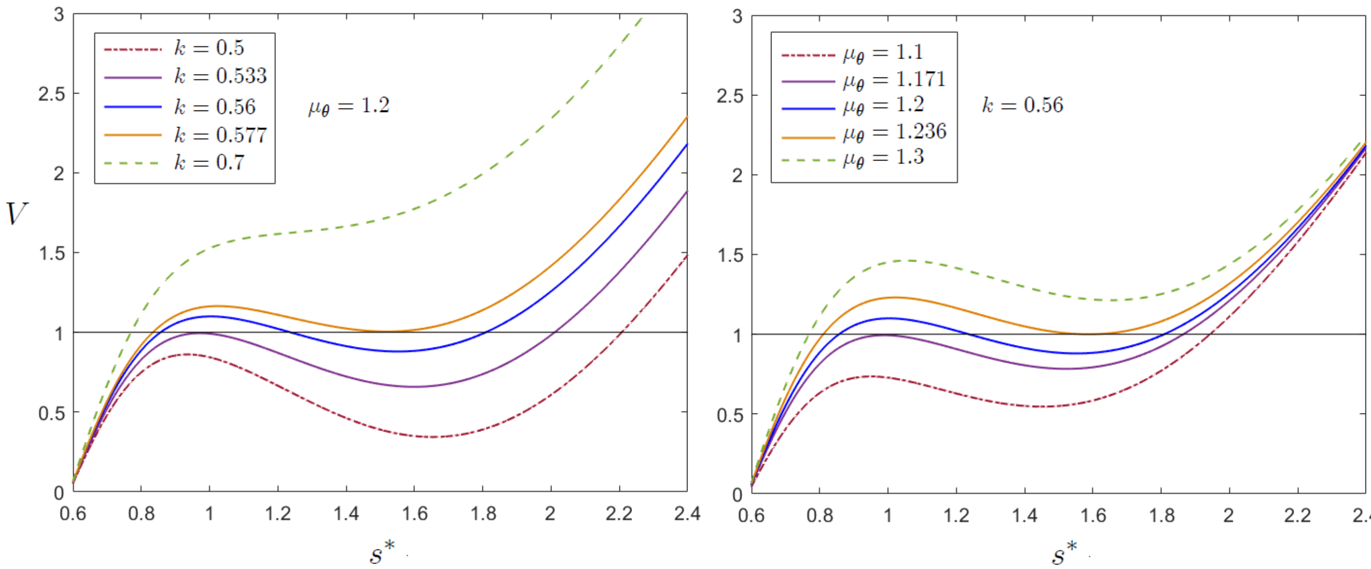


Figure 5. Equation $V(s^*;\mu_\theta,k)=1$.

Formally, Proposition 1 follows.

Proposition 1 ***(Banking system with an asset market)*** *Consider the limiting case of $\sigma_s\to 0$. When k is high enough or μ_θ is low enough, the creditor run-asset market equilibrium at $t=1$ in Lemma 3 is always unique; when k is not too high and μ_θ is not too low, multiple equilibria can exist for some values of (μ_θ,k).*

(Comparative statics) At a stable equilibrium, $\frac{\partial s^*}{\partial\mu_\theta}<0$ and $\frac{\partial s^*}{\partial k}<0$ (run threshold) together with $\frac{\partial(\varphi/k)}{\partial\mu_\theta}<0$ and $\frac{\partial(\varphi/k)}{\partial k}<0$ (fire-sale price discount).[15]

Even under the limit $\sigma_s\to 0$, multiple equilibria can exist *at the system level*. The intuition is as follows. The presence of a common asset market gives rise to strategic complementarities among creditors of different banks, in addition to the complementarities among creditors of the same bank. That is, there is an increased degree of strategic complementarity among creditors in the system, which makes equilibrium multiplicity more likely (see Appendix A for more details).

Based on Lemma 3 and Proposition 1, Figure 6 illustrates what happens if a shock hits μ_θ or k.[16] A small shock to μ_θ or k, triggering the feedback loop between s^* and φ, can lead to a large

[15] At a stable equilibrium, the slope of the best response function is lower than 1, that is, $\frac{\partial s^{*h}}{\partial s^*}<1$, where s^{*h} is the threshold used by an individual creditor and s^* is the threshold used by other creditors. At an unstable equilibrium, $\frac{\partial s^{*h}}{\partial s^*}>1$. This implies that at a stable (unstable) equilibrium, $\frac{\partial V(s^*;\mu_\theta,k)}{\partial s^*}>(<)0$, shown in Appendix A.

[16] Liu (2019) endogenizes the market liquidity k.

change in s^* and φ along a selected stable equilibrium. The existence of multiple equilibria implies an additional channel of amplification — multiplicity jumps. The right panel of Figure 6 illustrates the effect, where the curve $s^* = s^*(\varphi; k)$ is given by (8) and the curve $\varphi = \varphi(s^*, \mu_\theta)$ is given by (9).[17] A negative shock to μ_θ results in the equilibrium moving from A to B (through the feedback loop) or to B' (through the multiplicity jump). A negative shock to k has a similar effect.

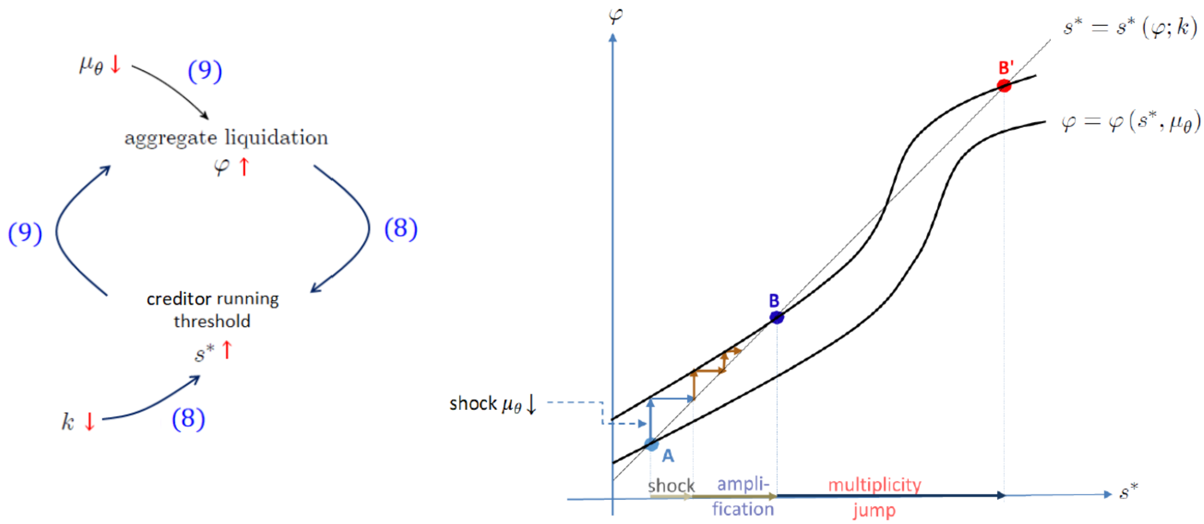


Figure 6. Amplification and multiplicity.

Before closing this section, we show a difference between within-bank externality and cross-bank externality in terms of the impact on run incentives. To formalize the idea, consider an individual bank. Suppose that for some reason (e.g., under constraints) a proportion λ_0 of bank i's creditors will choose to withdraw *for sure*. What then is the rollover threshold s^* for the other creditors of bank i? Similar to (8), s^* is given by

$$V(s^*; \lambda_0, \varphi) \equiv \frac{l_i - F\lambda_0}{F - F\lambda_0} \cdot \frac{D(s^*) - 1 + \Delta}{\Delta} = 1, \tag{11}$$

where $l_i = l_i(\theta_i = s^*) = s^* - \varphi/k$ by (2). Note that $\lambda_0 > 0$ reduces both the numerator and the denominator.

Lemma 4 *The externality among creditors of the same bank is characterized by* $\frac{\partial V(s^*;\lambda_0,\varphi)}{\partial \lambda_0} = \left(\underbrace{(-F)/(F - F\lambda_0)}_{(-)} + \underbrace{F(l_i - F\lambda_0)/(F - F\lambda_0)^2}_{(+)} \right) \frac{D(s^*)-1+\Delta}{\Delta} < 0$*, while the externality among creditors of different banks is characterized by* $\frac{\partial V(s^*;\lambda_0,\varphi)}{\partial \varphi} = \left(\underbrace{(-1/k)/(F - F\lambda_0)}_{(-)} \right) \frac{D(s^*)-1+\Delta}{\Delta} < 0.$

[17] Brunnermeier and Reis (2019) provide a review of amplification and multiplicity.

Lemma 4 illustrates that the externality among creditors of the same bank has two opposing forces while the externality among creditors of different banks has a single negative force.

II. Full Model with Liquid Asset Holdings

The full model in this section considers the setting in which banks hold liquid assets as well as illiquid assets. We show that interbank trading arises endogenously and the two types of trading — fire sales to outside investors and interbank trading — co-exist in the asset market. Three variables — creditor runs, fire-sale prices, and interbank rates (repo rates) — are jointly determined in equilibrium. We also examine how the distribution of liquidity across banks affects systemic bank runs.

In what follows, we study the equilibrium at $t = 1$ for given liquid asset holdings of banks. In Appendix B, we study the equilibrium at $t = 0$ to endogenize liquid asset holdings.

A. Equilibrium at $t = 1$ under Liquid Asset Holdings

We slightly modify the setup of the baseline model by instead assuming that at $t = 0$, the asset side of a bank's balance sheet is given by $(c, 1 - c)$, where c is the amount of liquid asset holdings (or simply "cash") and $1-c$ is the units of risky assets. The liability side is still given by $(F, 1-F)$. The rest of the setup remains the same as in the baseline model in Section I. Paralleling Section I.B.1, the equilibrium at $t = 1$ has three elements.

Aggregate liquidation in the system. Similar to (9), given the rollover threshold s^* of creditors for all banks, the total measure of bank assets under liquidation is given by

$$\varphi = (1 - c)\, \Phi \left(\frac{s^* - \mu_\theta}{\sigma_\theta} \right). \tag{12}$$

Asset market in equilibrium. Recalling the timeline in Figure 4, given that creditors of all banks use the rollover threshold s^*, we find the asset market equilibrium of the subgame. We will show that in equilibrium the structure of the asset market is as follows. In the asset market, banks are *endogenously* divided into three segments: $\theta_i \in (-\infty, s^{**}]$, (s^{**}, s^*), and $[s^*, +\infty)$, where s^{**} is another threshold (to solve). Weak banks with asset quality $\theta_i \in (-\infty, s^{**}]$ sell/repo their assets to strong banks $\theta_i \in [s^*, +\infty)$, while intermediate-quality banks $\theta_i \in (s^{**}, s^*)$ sell their assets to outside investors. Note that bank buyers are risk-neutral but potentially financially constrained,[18] while outside investors are risk-averse. Outside investors, as well as bank buyers, have perfect or almost perfect information about seller banks' asset quality.[19]

[18] In studying the equilibrium at $t = 0$, we will show banks optimally choose not to hold too much cash ex ante.

[19] As in the baseline model, we focus on the fully-revealing equilibrium or simply assume that bank buyers and outside investors receive private signals with diminishing noise about a seller bank's asset quality (i.e., $\sigma_x \to 0$) as in Morris and Shin (2004), similar to the precision of bank depositors' information $\sigma_s \to 0$. Empirical evidence in Afonso et al. (2011) shows that industry buyers of bank assets can have fairly precise information about asset quality.

Denote by η the aggregate fraction of liquidated assets that are absorbed by outside investors in equilibrium, by l_i the price of bank i's assets in selling to outside investors, and by I the expected return of interbank trading (equivalently, $1/I$ is the discount rate between $t = 1$ and $t = 2$).

(Step 1: Prices determine allocation). Lemma 1 implies asset prices offered by outside investors

$$l_i \equiv l(\theta_i) = \theta_i - (\eta\varphi)/k. \tag{13}$$

For bank buyers, who are price takers in the competitive asset market, they offer the price $\frac{\theta_i}{I}$ for bank i's assets. Therefore, on the seller side, a bank with asset quality θ_i chooses to sell to peer banks through interbank trading rather than to outside investors through fire sales if and only if $\frac{\theta_i}{I} \geq l(\theta_i)$. That is, there exists another threshold $\theta_i = s^{**}$ below which selling to peer banks is optimal and above which selling to outside investors is optimal, where s^{**} solves $\frac{\theta_i}{I} = l(\theta_i)$, or

$$I = \frac{s^{**}}{s^{**} - (\eta\varphi)/k}. \tag{14}$$

In other words, seller banks are *endogenously sorted* in the asset market: banks with asset quality $\theta_i \in (-\infty, s^{**}]$ choose to sell to peer banks while banks with asset quality $\theta_i \in (s^{**}, s^*)$ prefer selling to outside investors.[20] Intuitively, risk-neutral financially-constrained bank buyers seek the highest return (IRR) while CARA-utility deep-pocketed outside investors purchase the assets until the risk-adjusted NPV hits zero.

(Step 2: Allocation determines market clearing). Given that the outside investor sector absorbs bank assets of quality $\theta_i \in (s^{**}, s^*)$, it follows that $\eta = \left(\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right) - \Phi\left(\frac{s^{**}-\mu_\theta}{\sigma_\theta}\right)\right) \Big/ \Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$. Given that assets from banks with $\theta_i \in (-\infty, s^{**}]$ are absorbed by peer banks $\theta_i \in [s^*, +\infty)$, market clearing (akin to cash-in-the-market pricing) dictates

$$\begin{cases} \int_{\theta_i=-\infty}^{s^{**}} \frac{(1-c)\theta_i}{I} \cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right) = \int_{\theta_i=s^*}^{+\infty} c \cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right) & \text{if } s^{**} < s^* \\ \int_{\theta_i=-\infty}^{s^{**}} \frac{(1-c)\theta_i}{I} \cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right) \leq \int_{\theta_i=s^*}^{+\infty} c \cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right) & \text{if } s^{**} = s^* \end{cases}. \tag{15}$$

In (15), if s^{**} is an interior solution (i.e., $s^{**} < s^*$), the market-clearing condition is binding; if s^{**} is the corner solution (i.e., $s^{**} = s^*$), the total supply of liquidity can be excess.

Now we can explain and summarize two prices (returns) that individual banks face when they trade in the competitive asset market. First, in the case that a bank is short of liquidity, it needs to sell its risky assets in the asset market and the liquidation price is given by

$$\hat{l}_i \equiv \hat{l}(\theta_i) = \begin{cases} \frac{\theta_i}{I} & \text{when } \theta_i \in (-\infty, s^{**}] \\ l(\theta_i) = \theta_i - (\eta\varphi)/k & \text{when } \theta_i \in (s^{**}, s^*). \end{cases} \tag{16}$$

[20] A bank's owner maximizes the total value of the bank (its debt value plus equity value) when liquidating assets. Also, when σ_θ is sufficiently small relative to μ_θ, the probability of θ_i being negative is negligible.

Second, in the case that a bank has excess liquidity (i.e., $\theta_i \in [s^*, +\infty)$), it buys assets from other banks with expected return I and corresponding *risky* return

$$\tilde{I} = I + \frac{I}{\mathbb{E}\left(\theta_i | \, \theta_i \leq s^{**}\right)} e. \tag{17}$$

Considering that paying price $\frac{\theta_i}{I}$ entitles a risky payoff $\theta_i + e$ while the average θ_i in the interbank market is $\mathbb{E}\left(\theta_i | \, \theta_i < s^{**}\right)$, therefore, on average paying price $\frac{\mathbb{E}(\theta_i | \theta_i < s^{**})}{I}$ entitles a risky payoff $\mathbb{E}\left(\theta_i | \, \theta_i \leq s^{**}\right) + e$, which explains $\tilde{I}$. In our model, trading between banks in the asset market — essentially exchanging cash flows across time — can be interpreted as asset buying/selling or secured lending/borrowing, and hence the return I can also be interpreted as an interbank lending rate or a repo rate. Lemma 5 summarizes the asset market equilibrium with interbank trading.

Lemma 5 *The asset market in equilibrium, characterized by $(s^{**}, \{l_i\}, I)$ for a given (s^*, c), solves the system of equations (12) to (15). The equilibrium is unique, determining the asset liquidation prices given in (16) and the interbank return given in (17). The equilibrium has two cases.*

i) When c is high enough such that $\int_{\theta_i = s^*}^{+\infty} c \cdot d\Phi\left(\frac{\theta_i - \mu_\theta}{\sigma_\theta}\right) \geq (1-c) \int_{\theta_i = -\infty}^{s^*} \theta_i \cdot d\Phi\left(\frac{\theta_i - \mu_\theta}{\sigma_\theta}\right)$, the equilibrium is a corner solution in which peer banks absorb all of the liquidated assets and there is no fire sale to outside investors (that is, $s^{**} = s^*$, $l(\theta_i) = \theta_i$, and $I = 1$).

ii) When $c > 0$ is lower than the threshold in i), the equilibrium is an interior solution in which fire sales to outside investors and interbank trading co-exist.

Creditor run in equilibrium for an individual bank. Anticipating the subgame with the price rules given in (16) and (17), creditors make their rollover decisions in the first stage of the game at $t = 1$ by choosing threshold s^*. Paralleling (8), the creditor-run equilibrium is given by

$$\frac{c + (1-c) \cdot l\left(\theta_i = s^*\right)}{F} \cdot \frac{\hat{D}(s^*) - 1 + \Delta}{\Delta} = 1, \tag{18}$$

where function $l(\theta_i)$ is given in (16) and the debt value is redefined as $\hat{D}(\theta_i) \equiv \mathbb{E}\left(\min\left[R, \frac{(1-c)v_i + c\tilde{I}}{F}\right] |\theta_i\right)$, with $v_i \sim N(\theta_i, \sigma_e^2)$ and $\tilde{I}$ given in (17), as a survival bank has a payoff from interbank lending, $c\tilde{I}$.

Proposition 2 ***(Banking system with an asset market under liquid asset holdings)*** *With liquidity holdings, the creditor run-asset market equilibrium at $t = 1$, characterized by $(s^*, (s^{**}, \{l_i\}, I), \varphi)$ for a given (μ_θ, k, c), solves the system of equations (12) to (15) and (18) under the limit $\sigma_s \to 0$.*

We have the following comparative statics under certain mild regularity conditions: for a low c, at a stable equilibrium, $\frac{\partial s^*}{\partial k} < 0$ and $\frac{\partial s^*}{\partial c} < 0$; $\frac{\partial((\eta\varphi)/k)}{\partial k} < 0$ and $\frac{\partial((\eta\varphi)/k)}{\partial c} < 0$; $\frac{\partial I}{\partial k} < 0$ and $\frac{\partial I}{\partial c} < 0$.

B. Equilibrium at $t = 1$ under Heterogeneous Liquid Asset Holdings

In the previous subsection, we assume homogeneous liquid asset holdings and show that the aggregate level of liquid asset holdings, c, is a state variable in determining systemic bank runs. In this subsection, we study the equilibrium under heterogeneous liquid asset holdings and examine whether the *distribution* of liquidity holdings across banks matters for financial stability. The main purpose of this study is to lay the groundwork for deriving policy implications in Section IV.

We extend the baseline model by considering liquidity holdings of banks under heterogeneity. Specifically, assume that before creditors make their rollover decisions at $t = 1$, the asset side of bank i's balance sheet is given by $(c_i,\ 1)$ for some reason, where c_i is the amount of liquid asset (cash) holdings and 1 is the units of risky asset holdings. The short-term debt of a bank is still made up of depositors of mass F. Cash holdings c_i are distributed across banks according to a symmetric distribution with the fixed mean c in the support $[0, 2c]$. Denote the c.d.f. by $G(\cdot)$, which implies that

$$\int c_i dG(c_i) = c, \tag{19}$$

and in turn that the aggregate amount of cash holdings in the system is the fixed c. The baseline model is a special case with degenerate distribution $c = 0$. Also assume that the realization of θ_i for bank i is independent of its cash holdings c_i. That is, banks are heterogeneous along two dimensions (c_i, θ_i), where c_i is known to creditors and θ_i is not.

The setup above maps to the scenario in which the government decides to intervene and support banks at $t = 1$ by injecting liquidity into them (see Section IV). It is worth noting that we could alternatively choose the setting of banks' asset side being $(c_i, 1 - c_i)$ instead of $(c_i, 1)$. However, this alternative setting is less relevant. First, it adds another dimension of heterogeneity, namely, banks are heterogeneous not only in their cash holdings but also in their risky asset holdings. Second, more importantly, studying the setting $(c_i, 1 - c_i)$ is less relevant to policy analysis in Section IV.

Creditors of a bank make rollover decisions contingent on the bank's cash holdings. That is, the rollover strategy of bank i'creditors becomes $\left\{s_i^h, c_i\right\} \longmapsto \begin{cases} Call & s_i^h < s^*(c_i) \\ Hold & s_i^h \geq s^*(c_i) \end{cases}$, where $\left\{s_i^h, c_i\right\}$ is the information set and $s^*(c_i)$ is the rollover threshold. The equilibrium of the system at $t = 1$ is characterized by $(s^*(c_i), (s^{**}, \{l_i\}, I), \varphi)$ for a given $(\mu_\theta, k, G(\cdot))$, where $(s^{**}, \{l_i\}, I)$ is the equilibrium outcome of the asset market as defined in the previous subsection.

Equilibrium. The equilibrium is similar to that in the previous subsection. First, φ becomes

$$\varphi = \int \Phi\left(\frac{s^*(c_i) - \mu_\theta}{\sigma_\theta}\right) dG(c_i). \tag{20}$$

Second, as all banks are treated equally in the asset market, s^{**} is common (independent of c_i) and

Eq. (13)-(14) don't change, with η replaced by $\eta = \left(\int_{s^*(c_i)>s^{**}} \left[\Phi\left(\frac{s^*(c_i)-\mu_\theta}{\sigma_\theta}\right) - \Phi\left(\frac{s^{**}-\mu_\theta}{\sigma_\theta}\right) \right] dG(c_i) \right) / \varphi$. Note that if c_i-banks are with a particularly high c_i such that $s^*(c_i) < s^{**}$, those banks do not sell assets to outside investors and only sell to peer banks in the case of liquidation. The market-clearing condition (15) is replaced by

$$\int_{s^*(c_i)\leq s^{**}} \Gamma(s^*(c_i)) \frac{1}{I} dG(c_i) + \int_{s^*(c_i)>s^{**}} \Gamma(s^{**}) \frac{1}{I} dG(c_i) = \int c_i \left[1 - \Phi\left(\frac{s^*(c_i) - \mu_\theta}{\sigma_\theta} \right) \right] dG(c_i), \tag{21}$$

where $\Gamma(x) := \int_{\theta_i=-\infty}^{x} \theta_i d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right)$, the aggregate fundamental value of assets with quality up to x that are put on sale. The market-clearing condition is binding when c is not large as shown in Lemma 5. Third, the equation of creditor-run equilibrium (18) is replaced by

$$\frac{c_i + l(\theta_i = s^*(c_i))}{F} \cdot \frac{D(s^*(c_i)) - 1 + \Delta}{\Delta} = 1 \ , \tag{22}$$

where function $l(\theta_i)$ is given in (16) and debt function $D(\theta_i)$ is defined in (3). To prepare for studying policy implications, here we assume that bank depositors do not claim part of the payoff at $t = 2$ from their bank's interbank lending, if any (the payoff goes to the government; see Section IV). In other words, cash holdings affect the illiquidity risk but do not change the insolvency risk.

The system of equations (13), (14), and (19) to (22) solves the equilibrium. We are interested in the comparative statics regarding how the distribution $G(\cdot)$ for a given fixed mean c affects the aggregate inefficient fire sales $\eta\varphi$. To find out, we proceed as follows. The aggregate supply of liquidity from peer banks to the asset market, denoted by S^L, is given by

$$S^L = \int c_i \left[1 - \Phi\left(\frac{s^*(c_i) - \mu_\theta}{\sigma_\theta} \right) \right] dG(c_i), \tag{23}$$

while the aggregate fundamental value of bank assets put on sale (across *all* distressed banks), denoted by D^L, is given by

$$D^L = \int \Gamma(s^*(c_i)) dG(c_i).$$

We then define the *aggregate liquidity shortage* of the banking sector or equivalently the net amount of excess liquidity (which is negative) as

$$\Pi \equiv S^L - D^L = \int \Lambda(c_i) \, dG(c_i), \tag{24}$$

where

$$\Lambda(c_i) \equiv c_i \left[1 - \Phi\left(\frac{s^*(c_i) - \mu_\theta}{\sigma_\theta} \right) \right] - \Gamma(s^*(c_i)).$$

Proposition 3 *With heterogeneous liquid asset holdings, the creditor run-asset market equilibrium at $t = 1$ solves the system of equations (13), (14), and (19) to (22).*

(Comparative statics) Given a low c, when μ_θ or k is low, under a small enough σ_θ and certain mild regularity conditions, $\eta\varphi$ (inefficient fire sales) is lower under the distribution $G^{''}$ than under the distribution $G^{'}$, where $G^{'}$ second-order stochastically dominates $G^{''}$.

When the severity of a crisis is high (i.e., a low realization of μ_θ or k), an increase in dispersion of cash holdings can reduce inefficient fire sales in the system. Intuitively, for a crisis of high severity in which a large proportion of banks suffer runs, distributing the limited amount of aggregate cash holdings evenly across banks would not help as all banks would still have a low probability of survival. However, if the limited resources are concentrated to a smaller portion of banks, these banks would have a significantly increased chance of surviving. In fact, the survival probability of a c-bank is increasing and nonlinear in c (for low c and small enough σ_θ). In particular, once they survive, they would lend in the interbank market to help other banks. That is, the cash holdings for these receiving banks can be "reused": making them survive more likely and (re)entering the interbank market.

To elaborate, in the presence of the endogenous interbank market, the distribution of c_i affects the *aggregate* supply of liquidity in the interbank market from peer banks and hence the volume of inefficient fire sales to outside investors. More specifically, $\Lambda(c_i)$ is nonlinear and *convex* in c_i (in the range of a low c_i). The economic intuition is as follows. How much liquidity can a bank supply to the interbank market? It depends on two factors: whether the bank can survive (i.e., not suffer a run) and how much liquidity the bank possesses conditional on its survival. Both factors are a function of c_i. In fact, in $\Lambda(c_i)$, the term $1 - \Phi\left(\frac{s^*(c_i)-\mu_\theta}{\sigma_\theta}\right)$ corresponds to *the probability of survival* and the term c_i corresponds to *the amount of liquidity to supply conditional on survival.* The product of these two terms, $c_i\left[1 - \Phi\left(\frac{s^*(c_i)-\mu_\theta}{\sigma_\theta}\right)\right]$, is convex in c_i, so a more dispersed distribution $G(c_i)$ increases the aggregate supply of liquidity (by Jensen's inequality). Note that the force of the term $\Gamma(s^*(c_i))$ is dominated under a low μ_θ or k (a severe crisis) and a small enough σ_θ.

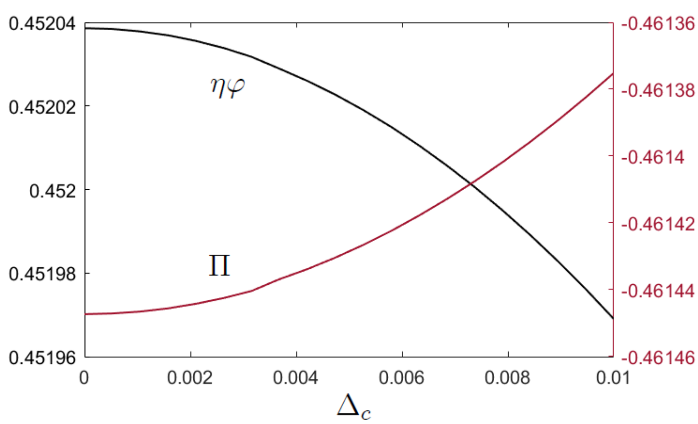


Figure 7. Aggregate inefficient fire sales ($\eta\varphi$) and aggregate liquidity shortage (Π) as a function of cash holding dispersion Δ_c.

Figure 7 gives an illustration based on a simulation exercise. We consider the simple binomial distribution of $G(\cdot)$, namely, $c_i \in \{c + \Delta_c, c - \Delta_c\}$ with a 50% probability for each realization, where an increase in $\Delta_c \in [0, c]$ corresponds to a mean-preserving spread. Figure 7 plots $\eta\varphi$ and Π in equilibrium as a function of Δ_c, under a set of parameter values $c = 0.01$, $\mu_\theta = 0.5$, $\sigma_\theta = 0.2$, $F = 0.6$, $k = 0.9$, $R = 1.1$, $\sigma_e = 0.2$, and $\Delta = 0.1$.

III. Full Model with Aggregate Uncertainty

In this section, we study the full model by considering aggregate uncertainty, that is, aggregate state μ_θ is no longer a constant but rather a stochastic variable. The central question we address in this section is whether aggregate uncertainty plays an important role in amplification mechanisms responsible for a systemic crisis, as many commentators emphasize uncertainty as a key factor in crisis episodes (e.g., Bernanke (2009), Gorton and Metrick (2010a), and Bloom et al. (2018)).

To derive economic insights, the full model needs to be solved analytically, which is challenging since one signal is used to infer two-layer, correlated fundamentals while the signal precision must also be taken to the limit for model tractability. The Laplacian property for the standard global-games setting that makes solving the equilibrium convenient cannot be directly applied to our setting. We find a new approach to solving the equilibrium.

A. Setting

Consider the setting as in Section I, but assume that the aggregate state μ_θ has the prior distribution $\mu_\theta \sim N(\bar{\mu}_\theta, \sigma^2_{\mu_\theta} = \tau^{-1}_{\mu_\theta})$, where $\bar{\mu}_\theta$ is a constant and common knowledge. As in the global-games literature, $\bar{\mu}_\theta$ can be interpreted as a public signal for the aggregate state μ_θ (see, e.g., Morris and Shin (2003)). The special case of $\sigma_{\mu_\theta} = 0$ corresponds to no aggregate uncertainty — the baseline model in Section I. Individual creditors' private information is still their private signal s_i^h, that is, an individual creditor's information set at $t = 1$ is now $\{\bar{\mu}_\theta, s_i^h\}$. Realistically, local information is more available and precise than global information.

B. Equilibrium

Denote by s^{*h} the threshold used by an individual creditor and by s^* the threshold used by other creditors of the same bank *as well as* by creditors of other banks. An individual creditor h knows the following two results (rules). First, given that all other creditors of the same bank as his use threshold s^*, the failure threshold of his bank, denoted by θ^*, is given by

$$\frac{\theta^* - \varphi/k}{F} = \Phi\left(\frac{s^* - \theta^*}{\sigma_s}\right). \tag{25}$$

This defines θ^* as a function of s^*, written as $\theta^* = \theta^*(s^*; \varphi)$. Second, because the creditors of all other banks use the same threshold s^*, the total measure of bank assets under fire sale in the system must be a function of s^* and μ_θ, written as

$$\varphi = \varphi\left(s^*, \mu_\theta\right), \tag{26}$$

which satisfies $\lim_{\sigma_s \to 0} \varphi\left(s^*, \mu_\theta\right) = \Phi\left(\frac{s^* - \mu_\theta}{\sigma_\theta}\right)$. Combining (25) and (26) yields one function $\theta^* = \theta^*\left(s^*; \varphi\left(s^*, \mu_\theta\right)\right)$, where the first argument s^* is the threshold used by other creditors of the same bank and the second argument s^* is the threshold used by creditors of other banks.

Individual creditor h does not know μ_θ and hence he does not know θ^*, but he does know the rule $\theta^* = \theta^*\left(s^*; \varphi\left(s^*, \mu_\theta\right)\right)$. Moreover, he does not know his bank's asset quality θ_i. However, his signal s_i^h reveals information about *both* θ_i *and* μ_θ. That is, one signal plays dual roles in Bayesian updating. His threshold s^{*h} is therefore given by the indifference condition

$$\mathbb{E}_{\mu_\theta, \theta_i | s_i^h}\left(\left(D(\theta_i) - 1\right) \cdot \mathbf{1}_{\theta_i \geq \theta^*} \middle| \, s_i^h = s^{*h}\right) = \mathbb{E}_{\mu_\theta, \theta_i | s_i^h}\left(\Delta \cdot \mathbf{1}_{\theta_i < \theta^*} \middle| \, s_i^h = s^{*h}\right), \tag{27}$$

where $\mathbb{E}_{\mu_\theta, \theta_i | s_i^h}(\cdot | s_i^h)$ is the conditional expectation operator and $\mathbf{1}_x$ is an indicator function defined in (5). The conditional distribution is now a joint distribution of two *dependent* variables (μ_θ, θ_i).

By symmetric equilibrium, we have

$$s^{*h} = s^*. \tag{28}$$

Lemma 6 *With aggregate uncertainty, the creditor run-asset market equilibrium at $t = 1$, characterized by $\left(s^*, \{l_i(\mu_\theta)\}, \varphi(\mu_\theta)\right)$ for a given $(\bar{\mu}_\theta, k)$, solves the system of equations (25) to (28). Under the limit $\sigma_s \to 0$, s^* solves the equation*

$$\left[\int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}\left(k\left(s^* - F\Phi\left(-z\right)\right)\right) - \frac{1}{1+\left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2}\left(\frac{s^* - \bar{\mu}_\theta}{\sigma_\theta}\right)}{\sqrt{\frac{\left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2}{1+\left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2}}}\right)\phi\left(z\right)dz\right]\frac{D(s^*) - 1 + \Delta}{\Delta} = 1; \tag{29}$$

for the extreme case $\sigma_{\mu_\theta} \to 0$ and a given σ_θ, equation (29) becomes

$$\left\{\frac{1}{F}\left[s^* - \Phi\left(\frac{s^* - \bar{\mu}_\theta}{\sigma_\theta}\right)\Big/ k\right]\right\}\frac{D(s^*) - 1 + \Delta}{\Delta} = 1;$$

for the other extreme case $\sigma_{\mu_\theta} \to +\infty$ (i.e., improper prior of μ_θ) and a given σ_θ, equation (29) becomes

$$\left[k\left(s^* - \frac{1}{2}F\right)\right]\frac{D(s^*) - 1 + \Delta}{\Delta} = 1.$$

We can see that when $\sigma_{\mu_\theta} \to 0$, equation (29) becomes the same as (10) (only with μ_θ replaced by $\bar{\mu}_\theta$), so it is possible for s^* to have multiple solutions as shown in Figure 5 and Proposition 1. In contrast, when $\sigma_{\mu_\theta} \to +\infty$, equation (29) clearly admits a unique solution.

Proposition 4 ***(Banking system with an asset market under aggregate uncertainty)*** *Consider the limiting case of $\sigma_s \to 0$. When σ_{μ_θ} is high enough (for a given σ_θ), the creditor run-asset market equilibrium at $t = 1$ in Lemma 6 is always unique, no matter the values of parameters $(\bar{\mu}_\theta, k)$. When σ_{μ_θ} is low enough, multiple equilibria can exist for some values of $(\bar{\mu}_\theta, k)$, as characterized in Proposition 1 (with μ_θ replaced by $\bar{\mu}_\theta$).*

With aggregate uncertainty there could be a unique equilibrium or multiple equilibria, depending on σ_{μ_θ} (for a given σ_θ and $\sigma_s \to 0$). The result of the amplification mechanism between creditor runs and asset prices always holds. In fact, with aggregate uncertainty, a feedback loop exists between s^* and the probability distribution of φ. A shock to $\bar{\mu}_\theta$ or k triggers the loop.[21]

With aggregate uncertainty, an individual creditor's private signal s_i^h contains information about the aggregate state or the status of other banks (besides the status of his own bank). When aggregate uncertainty increases, the information content of the private signal about the aggregate state rises. That is, the larger is σ_{μ_θ}, the more weight creditor h assigns to his private signal s_i^h, as opposed to the weight assigned to the prior $\bar{\mu}_\theta$, to infer the aggregate state μ_θ. Hence, upon receiving a given high signal s_i^h, under a larger σ_{μ_θ}, creditor h tends to believe more firmly that the amount of fire sales in the system is small (hence a high l_i), which decreases his incentive to run. In short, under a larger σ_{μ_θ}, an individual creditor h's threshold s^{*h} becomes less sensitive to the threshold s^* of other creditors, implying that equilibrium multiplicity is less likely.

C. Implications: Uncertainty Shocks and Amplification

Having solved the equilibrium with aggregate uncertainty, we now show amplification mechanisms that operate through uncertainty about the aggregate fundamentals (the second moment), other than through the fundamentals per se (the first moment).

C.1. Aggregate Uncertainty Amplifying Within-Bank Coordination

We show two channels of amplification. First, we show the posterior-mean channel. Recalling (25), in the system context, an individual bank i is essentially run by other banks via the asset market as well as by its own creditors, that is,

$$\theta^* = \underbrace{\Phi\left(\frac{s^* - \mu_\theta}{\sigma_\theta}\right)/k}_{\text{“run” by other banks via asset market}} + \underbrace{F\Phi\left(\frac{s_i^* - \theta^*}{\sigma_s}\right)}_{\text{run by own creditors}}, \tag{30}$$

where s_i^* and s^* denote the thresholds used by creditors of bank i and creditors of all other banks, respectively. In other words, in the system context, the interim illiquidity risk of an individual bank

[21] Based on (25) to (28), the equilibrium is essentially characterized by the fixed-point problem between s^* and the distribution of φ, given by the two equations $\mathbb{E}_{\mu_\theta,\theta_i|s_i^h}\left((D(\theta_i)-1)\cdot\mathbf{1}_{\theta_i \geq \theta^*(s^*;\varphi)}\middle| s_i^h = s^*\right) = \mathbb{E}_{\mu_\theta,\theta_i|s_i^h}\left(\Delta\cdot\mathbf{1}_{\theta_i < \theta^*(s^*;\varphi)}\middle| s_i^h = s^*\right)$ and $\varphi = \varphi(s^*, \mu_\theta)$.

comes from two parts: external market risk and internal coordination risk. Because μ_θ is stochastic, the first part is stochastic and individual creditors need to form their own expectations about it. Under certain conditions, an increase in uncertainty about μ_θ can cause the *marginal* creditors to adjust their expectation about μ_θ *downward* and hence their expectation about $\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$ upward in Bayesian updating, which precipitates their running at a higher threshold. That is, uncertainty about cross-bank "runs" exacerbates the within-bank coordination problem for every individual bank, triggering actual cross-bank "runs" with an amplification loop. Formally, we have the following comparative static result.

Corollary 1 *Consider the case in which the equilibrium is unique in Lemma 6. Under the sufficient condition that $\bar{\mu}_\theta$ is high enough and σ_{μ_θ} is low enough, the comparative statics $\frac{\partial s^*}{\partial \sigma_{\mu_\theta}} > 0$ for (29) follows. Hence, an increase in aggregate uncertainty σ_{μ_θ} leads to a larger proportion of banks suffering runs, $\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$, for any realized, unchanged aggregate fundamentals μ_θ.*

Corollary 1 implies that a small increase in aggregate uncertainty σ_{μ_θ} can result in a significant increase in s^* and thus a significant increase in $\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$ for any realized, *unchanged* aggregate fundamentals μ_θ. Figure 8 illustrates the effect, where the (best response) function $s_i^* = r\left(s^*; \sigma_{\mu_\theta}\right)$ is given by equation $V(s^*, s_i^*, \sigma_{\mu_\theta}) = 1$ in (A.10) in Appendix A.[22] A shock to σ_{μ_θ} triggers the feedback loop between s_i^* and s^*.

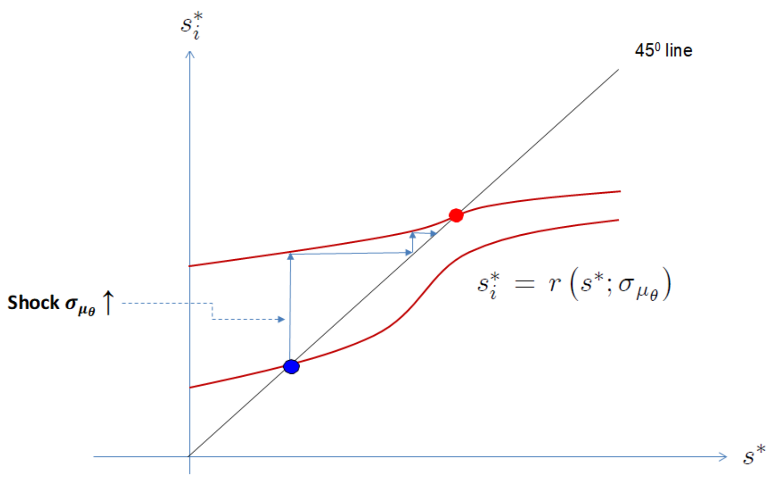


Figure 8. Effect of an increase in aggregate uncertainty σ_{μ_θ}.

Corollary 1 holds under the parameter conditions such that in equilibrium the proportion of banks failing in the system is less than 50% on average. Intuitively, without aggregate uncertainty

[22] In Appendix A, we show that the symmetric equilibrium in Lemma 6 can be alternatively characterized by the fixed-point problem between s_i^* and s^* with two equations: $V(s^*, s_i^*, \sigma_{\mu_\theta}) = 1$ in (A.10) and $s_i^* = s^*$. An increase in σ_{μ_θ} shifts up the curve $s_i^* = r\left(s^*; \sigma_{\mu_\theta}\right)$, but also lowers the slope $\frac{\partial s_i^*}{\partial s^*}$ at some relevant points (candidate equilibria) so equilibrium multiplicity is less likely.

(i.e., $\sigma_{\mu_\theta} \to 0$), the inference of μ_θ is simply equal to the prior $\bar{\mu}_\theta$. With aggregate uncertainty, the Bayesian inference of μ_θ for the *marginal* creditors who receive signal $s_i^h = s^*$ is a weighted average of the prior $\bar{\mu}_\theta$ and their signal $s_i^h = s^*$. As long as $s^* < \bar{\mu}_\theta$ (which implies $\Phi\left(\frac{s^* - \bar{\mu}_\theta}{\sigma_\theta}\right) < 0.5$ or the proportion of banks failing is less than 50% on average), those marginal creditors adjust their expectation about μ_θ *downward* (i.e., $\mathbb{E}\left(\mu_\theta \middle| s_i^h = s^*\right) < \bar{\mu}_\theta$), so those creditors would choose to run at a higher threshold. In other words, the equilibrium rollover threshold s^* is increasing in σ_{μ_θ}.

Next, we show the posterior-variance channel. Here we make an additional assumption: n is decreasing in φ, which essentially means that the supply of liquidity is negatively correlated with the demand for liquidity (in the same spirit as Liu (2019)). This assumption implies that $k \equiv n/\left(\gamma\sigma_e^2\right)$ is decreasing in φ, by recalling Lemma 1. For illustration and simplicity, assume that $n = \varphi^{-\beta}$ with $\beta > 0$, which implies $k = \varphi^{-\beta} k_0$, where $k_0 \equiv 1/\left(\gamma\sigma_e^2\right)$. Recalling (25), (30) is revised to

$$\theta^* = \underbrace{\left[\Phi\left(\frac{s^* - \mu_\theta}{\sigma_\theta}\right)\right]^{1+\beta} / k_0}_{\text{"run" by other banks via asset market}} + \underbrace{F\Phi\left(\frac{s_i^* - \theta^*}{\sigma_s}\right)}_{\text{run by own creditors}}. \tag{31}$$

Because $\beta > 0$, the first term in (31) is *convex* in μ_θ for a larger range of μ_θ when β is higher, while the convexity implies that the conditional expectation $\mathbb{E}\left(\left[\Phi\left(\frac{s^* - \mu_\theta}{\sigma_\theta}\right)\right]^{1+\beta} \middle| \mathcal{I}\right)$ increases when the information $\mathcal{I} = \left\{\bar{\mu}_\theta, s_i^h\right\}$ become noisier in the sense of a mean-preserving spread for the posterior distribution $\mu_\theta | \mathcal{I} \sim N\left(\frac{\tau_{\mu_\theta}}{\tau_{\mu_\theta} + \tau_\theta}\bar{\mu}_\theta + \frac{\tau_\theta}{\tau_{\mu_\theta} + \tau_\theta} s_i^h, \frac{1}{\tau_{\mu_\theta} + \tau_\theta}\right)$ under $\sigma_s \to 0$.

Corollary 2 *With the additional assumption that n is decreasing in φ, an increase in σ_{μ_θ} can lead to a higher s^* and hence a larger proportion of banks suffering runs, $\Phi\left(\frac{s^* - \mu_\theta}{\sigma_\theta}\right)$, for any realized, unchanged aggregate fundamentals μ_θ.*

With the additional assumption, the market depth is lower precisely when more fire sales occur, which implies that the fire-sale price discount (or the "run" by other banks via the asset market in (31)) is convex in the aggregate state μ_θ. Consequently, an increase in σ_{μ_θ}, which results in individual creditors being more uncertain about μ_θ, makes them form expectations of a deeper fire-sale price discount and thus precipitates their running at a higher threshold, with the feedback loop as in Figure 8. The macroeconomic literature on economic uncertainty makes similar convexity assumptions, for example, convexity is assumed in capital adjustment costs (Bachman and Bayer (2013), Bloom et al. (2018)), in search and matching frictions (Leduc and Liu (2016), Schaal (2017)), and in price stickiness in new Keynesian models (Basu and Bundick (2017)).

C.2. Shocks to Aggregate Uncertainty and to Bank-Level Dispersion

A large empirical literature documents that in bad economic times, both the uncertainty about the aggregate (macroeconomic) state and the firm- or bank-level dispersion increase (see, e.g., Altunbas, Manganelli, and Marques-Ibanez (2011), Bloom (2009), Bloom et al. (2018)). In particular, bank-level dispersion can rise more than aggregate uncertainty. Corollary 3 follows.

Corollary 3 *Consider the case in which there is a unique equilibrium initially in Lemma 6. When both σ_{μ_θ} and σ_θ increase and $\frac{\sigma_{\mu_\theta}}{\sigma_\theta}$ decreases, multiplicity (with a new less-efficient stable equilibrium) can emerge.*

Corollary 3 highlights another channel through which an increase in the second moment of fundamentals has an amplification effect — potential multiplicity jumps as depicted in Figure 6. In fact, based on (29), a tiny decrease in $\sigma_{\mu_\theta}/\sigma_\theta$ can cause a new (bad) stable equilibrium to emerge (while the equilibrium threshold s^* changes little for the original (good) stable equilibrium because of a tiny change in $\sigma_{\mu_\theta}/\sigma_\theta$). Intuitively, when bank-level dispersion increases more than does aggregate uncertainty, creditors' private signals about their own bank become less precise for inferring the status of other banks and hence these creditors have to rely more on the prior (public signal), although the prior also becomes less informative. In this case, self-fulfilling beliefs in response to the public signal have more room to form and multiplicity becomes more likely.

Remark. Before closing this section, we discuss how the effects of uncertainty shocks are related to the two-layer structure. First, that aggregate uncertainty has amplification effects in our model hinges crucially on the existence of an asset market (e.g., Corollary 2). In a one-layer structure with a single bank where there is no asset market and idiosyncratic uncertainty is also aggregate uncertainty, aggregate uncertainty has no such effects. Second, Corollary 3 shows the amplification mechanism that works through the change in aggregate uncertainty relative to bank-level dispersion. A single-layer model clearly does not have bank-level dispersion. Third, recalling that the equilibrium for an individual bank i is given by equations (4) and (5) and the prior of bank i's fundamentals is $\theta_i \sim N(\mu_\theta, \sigma_\theta^2)$, one may wonder whether the comparative statics $\frac{\partial s^*}{\partial \sigma_\theta} > 0$ holds for this single-bank equilibrium. It follows that $\frac{\partial s^*}{\partial \sigma_\theta} = 0$ under the limit $\sigma_s \to 0$. In contrast, for the two-layer structure, we have $\frac{\partial s^*}{\partial \sigma_{\mu_\theta}} > 0$ and $\frac{\partial s^*}{\partial \sigma_\theta} \lesseqgtr 0$ under the limit $\sigma_s \to 0$.

IV. Policy Implications

In this section, we analyze policy implications of our model, including ex post and ex ante policies. Our model framework enables us to derive some unique policy implications.

A. Ex Post Policies

A creditor run is inefficient ex post because it causes long-term illiquid assets to be prematurely liquidated and transferred to outside investors.[23] In particular, when a bad shock (such as the

[23] The short-term debt of a bank in our model may play a disciplining role (Calomiris and Kahn (1991), Diamond and Rajan (2001), and Eisenbach (2017)). For example, without the threat of a creditor run from short-term debt, the owner of a bank can take an (off-equilibrium) action that makes the bank asset riskier with a negative NPV.

realization of a low-probability state) hits μ_θ or k at $t = 1$, government intervention may be beneficial.

In the crisis of 2007 to 2009, the Federal Reserve adopted *unconventional* intervention measures, including injecting liquidity into financial institutions, creating emergency liquidity facilities for key credit markets, and directly purchasing long-term securities (Bernanke (2009)).

Using our framework, in this section we examine and compare two broad intervention measures: providing liquidity support to asset markets and injecting liquidity into banks. We analyze the pros and cons of each measure, demonstrate the tradeoff, and illuminate the optimal combination.

The timeline is as follows. At $t = 0$, the balance sheet of a bank is given.[24] Banks do not hold any liquid assets at $t = 0$ as in the baseline model. At $t = 1$, the aggregate state (shock) μ_θ and the idiosyncratic fundamentals $\{\theta_i\}$ are realized. Creditors of bank i perfectly observe μ_θ and receive a noisy signal about θ_i. The government observes the aggregate state μ_θ and may decide to intervene. Specifically, before creditors make their rollover decisions, the government can use its limited resources Q to support the banks, the asset market, or both. We make a weak assumption that the government has no information about individual fundamentals $\{\theta_i\}$.

A.1. Supporting the Asset Market

Ggovernment support of the asset market, in the form of directly or indirectly purchasing assets, can reduce the fire-sale pressure to outside investors and hence boost asset prices, which in turn makes creditors have less incentive to run in the first place. That is, supporting the asset market helps break the amplification loop between lower asset prices and more creditor runs.

Suppose the government uses liquidity in amount Q to support the asset market. Consistent with practice, trading among private agents (who have expertise and information) in the asset market establishes prices, and given the prices the government uses up the amount Q of liquidity to buy assets. This is equivalent to the case in which the government and private investors start a joint (co-investment) program/venture to purchase assets, or in which the government indirectly purchases part of the assets bought by private investors (this idea of asset purchases was initiated by Henry Paulson, the then U.S. Secretary of the Treasury, and later implemented by the Fed; see Bernanke (2015)).

The equilibrium is similar to that in Lemma 3, but with the government's involvement added. Concretely, the creditor run-asset market equilibrium at $t = 1$ is characterized by $(s^*, (\alpha, \{l_i\}), \varphi)$ for a given (μ_θ, k, Q), where the new term α denotes the aggregate fraction of liquidated assets absorbed by outside investors (with the remaining $1 - \alpha$ fraction absorbed by the government's

[24] See Appendix B for the study of the ex ante problem of endogenizing the balance sheet.

liquidity in equilibrium). The equilibrium solves the system of equations

$$\begin{array}{lll} l_i = \theta_i - (\alpha\varphi)/k & \text{(asset prices of secondary market)} & (a) \\ \frac{l_i(\theta_i = s^*)}{F} \cdot \frac{D(s^*)-1+\Delta}{\Delta} = 1 & \text{(creditor run of an individual bank)} & (b) \\ \varphi = \Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right) & \text{(aggregate liquidation in the system)} & (c) \\ (1-\alpha)\int_{\theta_i=-\infty}^{s^*} l_i(\theta_i) \cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right) = Q, & \text{(government's support for market)} & (d) \end{array} \quad (32)$$

where debt function $D(\theta_i)$ is defined in (3).

As an illustration, Figure 9 plots the equilibrium equation $V(s^*; Q) = 1$, where $V(s^*; Q)$ is defined similarly to (10), under a set of parameter values $\mu_\theta = 1.15$, $\sigma_\theta = 0.42$, $F = 0.6$, $k = 0.56$, $R = 1.1$, $\sigma_e = 0.2$, and $\Delta = 0.1$. We can see that a higher Q not only shifts the curve $V(s^*; Q)$ up but also reduces the curvature, making multiple equilibria less likely.

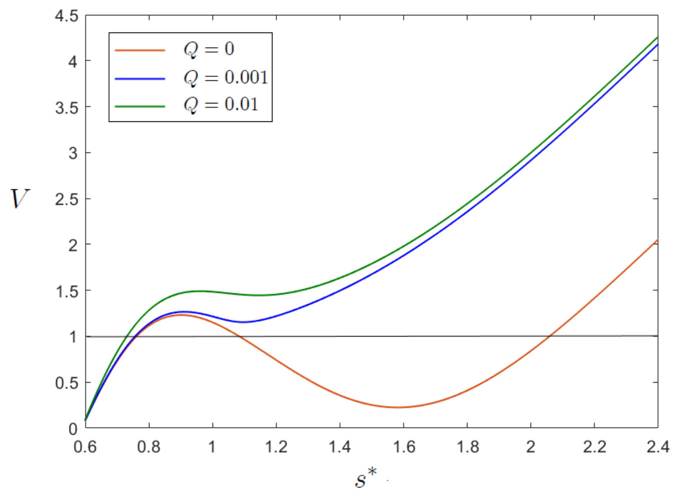


Figure 9. Equation $V(s^*; Q) = 1$.

Lemma 7 *Supporting the asset market (program (32)) de facto increases the market depth k, so it not only reduces the run threshold s^* for a given selected stable equilibrium, but can also eliminate equilibrium multiplicity.*

Supporting the asset market is effectively a price-contingent bailout: when asset prices are lower, the government is able to buy more distressed assets; to reduce fire-sale pressure to outside investors by the same level, the government would alternatively need to give banks a bigger bailout in the form of providing loans to them. Supporting the asset market is particularly effective in containing severe expectations-driven systemic runs: when a bad expectation about asset prices precipitates creditors running at a higher s^*, driving down asset prices, precisely at that time government liquidity, with a fixed aggregate amount, is able to buy more assets, which thus provides a cushion that dampens the decline in asset prices, so a bad expectation may not arise in the first place.

That is, the government's liquidity support to the asset market de facto improves market depth (the effective market depth can be expressed as $\hat{k} \equiv k/\alpha > k$ based on (32a)).[25]

A.2. Supporting Banks

Despite having no information about the asset quality of individual banks, the government can also choose to inject liquidity directly into banks. Importantly, after obtaining liquidity from the government, banks can trade in the interbank market, so the government's liquidity can flow where it is most needed. This policy can be broadly interpreted as a system-wide credit policy or even monetary policy.

Suppose the government decides to inject the amount Q of liquidity into banks and distribute it *evenly* across banks. Then, after the liquidity injection and before the rollover decisions of creditors at $t = 1$, the asset side of a bank's balance sheet is given by $(c, 1)$, where $c = Q$ is the amount of cash holdings and 1 is the units of risky assets. Under liquidity injection, the creditor run-asset market equilibrium at $t = 1$, characterized by $(s^*, (\eta, s^{**}, \{l_i\}, I), \varphi)$ for a given (μ_θ, k, Q), solves

$$\begin{aligned}
&l_i = \theta_i - (\eta\varphi)/k && \text{(asset prices of secondary market)}\\
&\tfrac{c+l_i(\theta_i=s^*)}{F} \cdot \tfrac{D(s^*)-1+\Delta}{\Delta} = 1 && \text{(creditor run of an individual bank)}\\
&\varphi = \Phi\left(\tfrac{s^*-\mu_\theta}{\sigma_\theta}\right) && \text{(aggregate liquidation in the system)}\\
&\eta\varphi = \Phi\left(\tfrac{s^*-\mu_\theta}{\sigma_\theta}\right) - \Phi\left(\tfrac{s^{**}-\mu_\theta}{\sigma_\theta}\right) && \text{(assets sold to secondary market)}\\
&I = \tfrac{s^{**}}{s^{**}-(\eta\varphi)/k} && \text{(interbank rate)}\\
&\textstyle\int_{\theta_i=-\infty}^{s^{**}} \tfrac{\theta_i}{I} \cdot d\Phi\left(\tfrac{\theta_i-\mu_\theta}{\sigma_\theta}\right) = \int_{\theta_i=s^*}^{+\infty} c \cdot d\Phi\left(\tfrac{\theta_i-\mu_\theta}{\sigma_\theta}\right) && \text{(interbank market clearing)}\\
&c \cdot 1 = Q, && \text{(government's injection into banks)}
\end{aligned} \tag{33}$$

where $D(\theta_i)$ is defined in (3). For simplicity, we have assumed the following payoff structure for the government in supporting banks. If a receiving bank suffers a run, it fails and the government gets nothing back; if a receiving bank survives and hence lends out the government's support liquidity in the interbank market, the government obtains the full claims to the interbank lending.[26] Essentially, the government provides non-recourse loans to banks (see the evidence in Duygan-Bum et al. (2013), among others).

The equilibrium given in (33) is essentially that in Proposition 2; the only differences are that banks' asset side is replaced by $(c, 1)$ and c is determined by the government's liquidity injection.

[25] The simple intuition is the following. Consider the asset price equation given in Lemma 1: $l = \theta - \frac{\varphi}{k}$ (with subscript i removed for simplicity). The market depth corresponds to the slope $\frac{dl}{d\varphi} = -\frac{1}{k}$. With liquidity support Q, the asset price can be intuitively written as $l = \theta - \frac{\varphi - \frac{Q}{l}}{k}$, where $\frac{Q}{l}$ is the amount of assets that the government purchases and clearly $\frac{Q}{l}$ increases when l decreases. It then follows that $\frac{dl}{d\varphi} = -\frac{1}{\hat{k}}$, where $\hat{k} = k + Q\frac{1}{l^2} > k$.

[26] Alternatively, we can assume that the government and the bank equityholder share the claims to the interbank lending, in which case the model algebra does not change at all. As long as bank depositors do not claim part of the interbank lending, the interbank return $\tilde{I}$ defined in (17) does not enter $D(s^*)$, which keeps the analysis clean.

How cash holdings of banks affect the equilibrium (including multiplicity) is shown in Proposition 2 and discussed around Figure A1 in Appendix A.

Lemma 8 *Supporting banks by injecting liquidity into them (program (33)) reduces the run threshold s^* for a given selected stable equilibrium.*

Two channels of mechanisms are at work for the result in Lemma 8. First, each bank has more cash and hence creditors have less incentive to run. Second, the liquidation values of banks' illiquid assets are pushed up and creditors have even less incentive to run. The second channel arises because part of the government's liquidity enters the interbank market and hence the fire-sale pressure to outside investors is alleviated and the market depth is in effect improved.

A.3. Optimal Combination of Intervention Measures

Given that the government has limited resources with which to support the system, it is important to study the optimal strategy for the government in the intervention. Our model formally characterizes the pros and cons of each measure, and thus allows us to study the optimal combination of the two measures. In this subsection, we first present the constrained optimization problem for the government and then analyze the tradeoff and show the optimal combination.

Recall that the government has an aggregate amount Q of liquidity to support the system. Suppose a portion with an amount Q_1 is employed to support the secondary asset market as in Section IV.A.1 and the remaining portion with an amount Q_2 is injected directly into the banks as in Section IV.A.2, where $Q_1 + Q_2 = Q$. The government is to choose the optimal allocation, (Q_1, Q_2), for a given Q. We proceed in two steps. First, we solve the creditor run-asset market equilibrium for a given (Q_1, Q_2). Second, we find the optimal allocation (Q_1, Q_2) for a given Q.

In the first step, given the intervention combination (Q_1, Q_2), the creditor run-asset market equilibrium at $t = 1$, characterized by $(s^*, (\eta, \alpha, s^{**}, \{l_i\}, I), \varphi)$ for a given $(\mu_\theta, k, Q_1, Q_2)$, solves

$$\begin{aligned}
&l_i = \theta_i - (\alpha\eta\varphi)/k && \text{(asset prices of secondary market)}\\
&\frac{c + l_i(\theta_i = s^*)}{F} \cdot \frac{D(s^*) - 1 + \Delta}{\Delta} = 1 && \text{(creditor run of an individual bank)}\\
&\varphi = \Phi\left(\frac{s^* - \mu_\theta}{\sigma_\theta}\right) && \text{(aggregate liquidation in the system)}\\
&\eta\varphi = \Phi\left(\frac{s^* - \mu_\theta}{\sigma_\theta}\right) - \Phi\left(\frac{s^{**} - \mu_\theta}{\sigma_\theta}\right) && \text{(assets sold to secondary market)}\\
&(1-\alpha)\int_{\theta_i = s^{**}}^{s^*} l_i(\theta_i) \cdot d\Phi\left(\frac{\theta_i - \mu_\theta}{\sigma_\theta}\right) = Q_1 && \text{(government's support for market)}\\
&I = \frac{s^{**}}{s^{**} - (\alpha\eta\varphi)/k} && \text{(interbank rate)}\\
&\int_{\theta_i = -\infty}^{s^{**}} \frac{\theta_i}{I} \cdot d\Phi\left(\frac{\theta_i - \mu_\theta}{\sigma_\theta}\right) = \int_{\theta_i = s^*}^{+\infty} c \cdot d\Phi\left(\frac{\theta_i - \mu_\theta}{\sigma_\theta}\right) && \text{(interbank market clearing)}\\
&c \cdot 1 = Q_2, && \text{(government's injection into banks)}
\end{aligned} \tag{34}$$

where debt function $D(\theta_i)$ is defined in (3). The equilibrium given in (34) is essentially the combination of the ones given in (32) and (33); in fact, (32) and (33) correspond to the two extreme cases $Q_2 = 0$ and $Q_1 = 0$, respectively. Notably, of the aggregate liquidation φ (in quantity), a portion with quantity $(1-\eta)\varphi$ is sold via interbank trading and the remainder with quantity $\eta\varphi$ is sold through the secondary market; of the remainder, a proportion α in quantity or value is absorbed by outside investors and the rest $1-\alpha$ is absorbed by the government's liquidity support. Figure 10 gives an illustration.

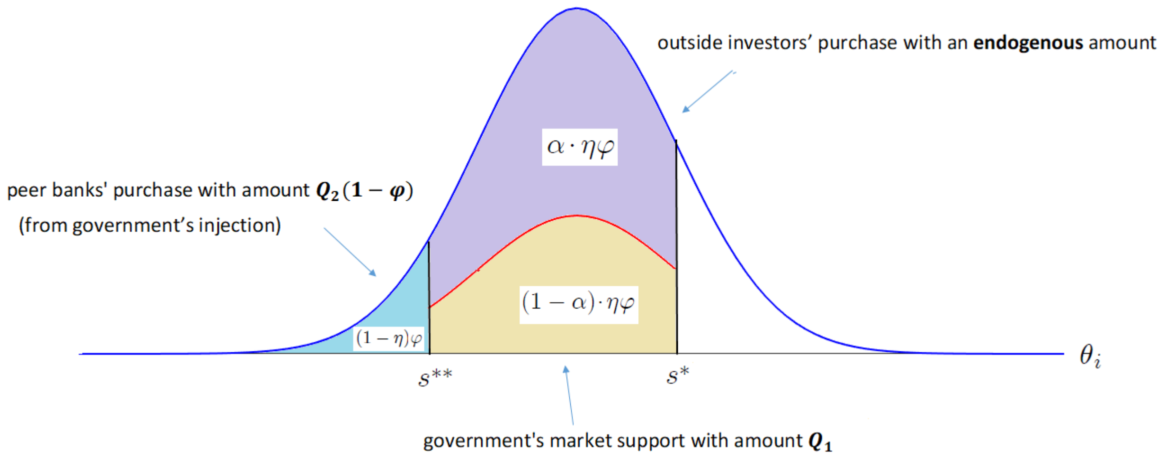


Figure 10. Where the liquidated assets with quantity φ go.

In the second step, the government's optimal allocation (Q_1, Q_2) is given by

$$\max_{Q_1,Q_2} Y(Q_1,Q_2) \equiv \left[\begin{array}{l} \int_{\theta_i=-\infty}^{s^{**}} (\theta_i + c)\cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right) + \int_{\theta_i=s^*}^{+\infty} (\theta_i + c)\cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right) \\ + \left(\begin{array}{l} \alpha \int_{\theta_i=s^{**}}^{s^*} \left[\left(\theta_i - \frac{\alpha\eta\varphi}{k}\right) + c\right]\cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right) \\ +(1-\alpha)\int_{\theta_i=s^{**}}^{s^*} (\theta_i + c)\cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right) \end{array} \right) \end{array} \right]$$

$$\text{s.t.} \quad (34) \quad \text{and} \quad Q_1 + Q_2 = Q. \tag{35}$$

In program (35), the objective function is to maximize the aggregate value for the entire banking sector and the government. Banks with $\theta_i \in (-\infty, s^{**}]$ fail but their illiquid assets are sold (transferred) to other banks, so there is no social efficiency loss, that is, the total value for different final claimants for such a bank is still the fundamental value $\theta_i + c$. Similarly, there is no social efficiency loss for banks with $\theta_i \in [s^*, +\infty)$. However, banks with $\theta_i \in (s^{**}, s^*)$ suffer inefficient fire sales, and part of their illiquid assets are sold and transferred to lower-valuation outside investors at the discounted prices $l_i = \theta_i - (\alpha\eta\varphi)/k$.[27] Note that in (35) we assume that the government cares only about the welfare of the banking sector. We can alternatively assume that the government also gives

[27] We can alternatively assume that, like the outside investors, the government is risk-averse. In this case, the model result does not change qualitatively.

some weight to the welfare of outside investors, considering that outside investors derive some (consumer) surplus in buying the assets. In this case, because the surplus for the outside investor sector is given by $n\mathbb{E}\left[U(W^j)|\{l_i\}\right] = n\mathbb{E}\left[-\exp\left(-\gamma\left[\int q_i(v_i - l_i)di\right]\right)\right] = -n\exp\left(-\frac{1}{2}(\gamma/k)\frac{1}{n}(\alpha\eta\varphi)^2\right)$ (see the proof in Appendix A), including this term in the objective function would not qualitatively change the optimization result in Proposition 5 below when γ is high enough.

Now we compare the mechanism behind the two intervention measures, listed in Figure 11.

Channels of mechanism \ Measures		**Inject liquidity into banks**	**Support secondary asset market**
Increasing liquidity holdings of individual banks (Channel 1)		Yes	No
Increasing liquidation value of illiquid assets of individual banks (in the case of a run)	Amount of liquidity entering the asset /interbank market (Channel 2)	1-φ proportion (from survival banks only)	100%
	Effectiveness of buying distressed assets (Channel 3)	More effective (buy most-distressed assets)	Less effective (buy less-distressed assets)

Figure 11. Comparison of mechanism behind intervention measures in affecting run incentive (s^*).

The objective function in (35) implies that the government intends to reduce the run threshold s^* (and consequently reduce aggregate inefficient fire sales $\alpha\eta\varphi$). Based on the creditor-run equilibrium equation in (34), there are two ways to do this: increase liquid asset holdings c and increase the liquidation value l_i of illiquid assets of individual banks, which give three channels of mechanism in Figure 11.

Supporting the asset market over supporting banks has advantages in Channel 2 but disadvantages in Channels 1 and 3. The comparison in Channel 1 in Figure 11 is easy to understand. To understand the comparison in Channels 2 and 3, based on (34) we calculate the quantity of distressed assets effectively purchased with the government's liquidity Q_1, which is given by

$$(1-\alpha)\,\eta\varphi = \frac{Q_1}{\mathbb{E}\left(l_i\left(\theta_i\right)|\theta_i \in \left(s^{**}, s^*\right)\right)}, \qquad \text{(via the support for market)} \tag{36}$$

where the denominator is the average price of the purchased assets. Note that outside investors purchase assets with quality in the range $\theta_i \in (s^{**}, s^*)$. Similarly, we calculate the quantity of distressed assets effectively purchased with the government's liquidity Q_2, which is given by

$$(1-\eta)\,\varphi = \frac{\int_{\theta_i=s^*}^{+\infty} c \cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right)}{\left(\int_{\theta_i=-\infty}^{s^{**}} \frac{\theta_i}{I}\cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right)\right)/\left(\left(1-\eta\right)\varphi\right)} = \frac{Q_2\left(1-\varphi\right)}{\mathbb{E}\left(\theta_i\,|\theta_i \le s^{**}\right)/I}, \quad \text{(via the support for banks)} \tag{37}$$

where the numerator is the aggregate liquidity from the surviving banks $\theta_i \in [s^*, +\infty)$, which enters the interbank market, and the denominator is the average price of the purchased assets. Note that interbank trading purchases assets with quality in the range $\theta_i \leq s^{**}$.

In comparing (36) and (37), we see the difference. Supporting the asset market means that all of the government's support liquidity enters the asset market, whereas supporting banks means that only part of the government's liquidity — the part that ends up in surviving banks — enters the interbank market. This is reflected in the difference between the two numerators. On the other hand, for the same amount of liquidity entering the asset/interbank market, the liquidity for interbank trading is more efficient in tackling the asset market stress than is the liquidity for outside investors. This is because the former purchases the most distressed assets $\theta_i \in (-\infty, s^{**}]$ while the latter purchases the less-distressed assets $\theta_i \in (s^{**}, s^*)$. In fact, comparing the two denominators, $\mathbb{E}\left(l_i\left(\theta_i\right)|\theta_i \in (s^{**}, s^*)\right) > \mathbb{E}\left(\theta_i \left|\theta_i \leq s^{**}\right.\right)/I$ holds. The "quantity–quality (efficiency)" tradeoff above can be alternatively illustrated by asking what the government's payoff is if it gives \$ 1 to the market versus to banks. The payoff is $1 \cdot \frac{\mathbb{E}(\theta|\theta_i \in (s^{**}, s^*))}{\mathbb{E}(l_i(\theta_i)|\theta_i \in (s^{**}, s^*))}$ versus $(1-\varphi)\cdot I$, where the two returns clearly satisfy $\frac{\mathbb{E}(\theta|\theta_i \in (s^{**}, s^*))}{\mathbb{E}(l_i(\theta_i)|\theta_i \in (s^{**}, s^*))} < I$.[28]

Under either intervention, the government's support liquidity Q will enter the banking system and in the end enter the failure banks which sell assets; see Figure 10. However, the two intervention measures have different implications for the distribution of Q among failure banks and for the amount of outside liquidity entering the banking system (which is endogenous).

Proposition 5 *Given a low Q, the government's optimal allocation of liquidity, (Q_1, Q_2), in supporting the system is given by program (35). There exist the following cases.*

i) When (μ_θ, k) is in the region such that in equilibrium the proportion of banks suffering runs in the system, $\varphi = \Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$, is high enough, the optimal allocation is $(Q_1, Q_2) = (Q, 0)$.

ii) When (μ_θ, k) is in the region such that $\varphi = \Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$ is low enough, the optimal allocation is $(Q_1, Q_2) = (0, Q)$.

iii) When (μ_θ, k) is in the region such that $\varphi = \Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$ is at an intermediate level, the optimal allocation is (Q_1, Q_2) with $Q_1 > 0$ and $Q_2 > 0$.

Proposition 5 implies that for a very severe (grave) crisis (case i), it is optimal for the government to use its limited resources to support the asset market only; for a mild crisis (case ii), supporting banks only is optimal; and for a severe crisis (case iii), it is optimal to mix support to banks and support to the asset market. Figure 12 provides an illustration based on a simulation exercise, where $Y(Q_1, Q_2)$ is the objective function in (35) and the parameter values are the same as in Figure 7, that is, $Q = 0.01$, $\sigma_\theta = 0.2$, $F = 0.6$, $k = 0.9$, $R = 1.1$, $\sigma_e = 0.2$, and $\Delta = 0.1$.

[28] Outside our model, as long as bank buyers are more efficient than outside investors in "arbitrage" (due to preference, information, expertise, or other frictions), the tradeoff identified here likely still holds.

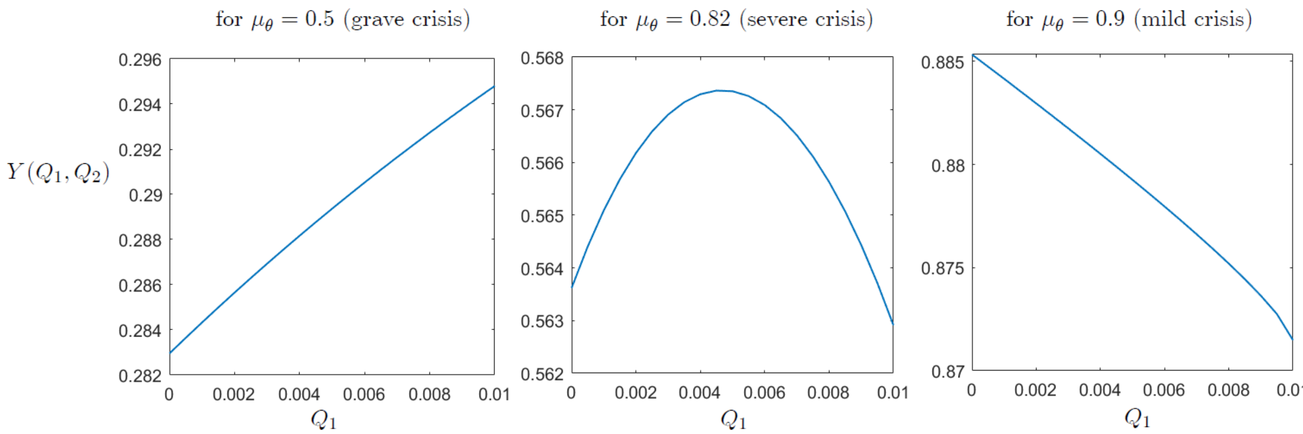


Figure 12. Optimal combination of intervention measures.

We explain the economic intuition behind Proposition 5, beginning with the corner solutions for cases i) and ii). When the crisis is very severe (with a large proportion of banks failing and deeply-discounted asset prices), supporting the asset market is optimal. In fact, in this scenario, survival banks are few (i.e., $1 - \varphi$ is low), so supporting banks would result in very little liquidity entering the interbank market — this disadvantage in "quantity" (Channel 2) far exceeds its advantage in "efficiency" (Channel 3). Moreover, the advantage in Channel 1 of supporting banks is dwarfed, considering that supporting the asset market is a "price-contingent" bailout: when the asset prices are low, the government's liquidity support can purchase a large amount of assets, so at that moment supporting the asset market is much more effective.[29] In contrast, when the crisis is mild, survival banks abound ($1-\varphi$ is high), so the disadvantage in "quantity" of supporting banks is not much and is outweighed by its advantage in "efficiency" (Channel 3) together with its advantage in Channel 1.

We turn next to the interior solution for case iii). An interior solution exists because of the "diminishing returns" effect of supporting the asset market (i.e., $\frac{\partial Y}{\partial Q_1}$ is decreasing in Q_1), given that a higher Q_1 pushes up the prices and reduces its own *marginal* effect. Note that the marginal effect $\frac{\partial Y}{\partial Q_2}$ is less sensitive to Q_2 (than $\frac{\partial Y}{\partial Q_1}$ is to Q_1), as Channel 1 of the mechanism does not depend directly on the asset prices or the aggregate state. Overall, equating the two marginal effects, $\frac{\partial Y}{\partial Q_1} = \frac{\partial Y}{\partial Q_2}$, yields an interior solution of Q_1.

So far we have assumed that, in the case of providing liquidity support to banks, the government distributes the liquidity evenly across banks. Based on Section II.B, when the crisis is of high severity (i.e., a low realization of μ_θ or k), an increase in the dispersion of cash holdings can improve

[29]To isolate the effect via Channel 1 of supporting banks, we can consider the policy in Section IV.A.2 but shut down the interbank trading, in which case the equilibrium given in program (33) is revised and the illiquidity risk corresponds to $c + l_i(\theta_i) = \theta_i - \varphi/k + Q$. In contrast, the effect of supporting the market is characterized by program (32), in which the illiquidity risk corresponds to $l_i = \theta_i - (\alpha\varphi)/k = \theta_i - \varphi/k + \frac{Q}{\bar{l}}/k$, where $\bar{l} = E(l_i(\theta_i)|\theta_i < s^*)$. Clearly, when the crisis is very severe, supporting the market is more effective.

efficiency. Now let us consider the policy of providing differential support to banks. Specifically, we consider a simple form of differential support: the government, despite having no information about the asset quality of individual banks, randomly gives half of the banks more liquidity and the other half less liquidity, that is, there are two types of banks (type-A and type-B) with $c_i \in \left\{c^A,\ c^B\right\}$, where $\frac{1}{2}\left(c^A + c^B\right) = Q$. Under this policy, the equilibrium is as given in Section II.B.

Corollary 4 *Given a low Q, when the crisis is severe enough (i.e., a low μ_θ or k) and supporting the asset market only is optimal in Proposition 5 (case i), the alternative policy of providing differential support to banks is less effective.*

Providing differential liquidity support to banks is more effective than providing equal liquidity support to banks when the crisis is of high severity, as shown in Proposition 3. However, it is still less effective than supporting only the asset market in such circumstances. The simulation results in Figures 7 and 12 illustrate. Under the same set of parameter values, Figure 7 implies that providing differential support to banks achieves the maximum value of the objective function given in (35) as $Y = \mu_\theta + c - (\eta\varphi)^2/k = 0.283$ at $c_i \in \left\{c^A = 0.02,\ c^B = 0\right\}$; in contrast, the maximum value of the objective function in the left panel of Figure 12 is $Y = 0.294$.

The policy of providing differential support to banks overcomes, to some extent, the weakness of the policy of providing equal support to banks (Channel 2 in Figure 11). However, when the crisis is severe enough, even type-A banks with a higher level of cash holdings c^A have a low probability of surviving and thus supporting banks would still result in very little liquidity entering the interbank market. Directly supporting the asset market is more effective in such circumstances.

B. Ex ante Policies

Basel III introduces requirements on liquid asset holdings, strengthening the requirements from the Basel II standard on banks' minimum capital (leverage) ratios (see Basel Committee on Banking Supervision (2011)). Our model provides a rationale for the new regulation.

Specifically, we endogenize liquid asset holdings at $t = 0$ in our model, study the optimal level of liquidity holdings for banks, and address the question of whether individual banks' choice is socially optimal. We show that individual banks' optimal level of liquid asset holdings in the decentralized equilibrium is typically lower than the constrained social optimum, which gives a rationale for the regulation on liquid asset holdings. To save space, the details are relegated to Appendix B.

As for banks' capital (leverage) ratios highlighted in Basel II, Eisenbach (2017) studies banks' optimal choice of the short-term debt level as market discipline in general equilibrium, which complements our work addressing banks' liquid asset holdings.

V. Conclusion

This paper presents a tractable framework of bank runs in the market-based banking system. In the system context, a run on one bank is affected by, and affects, runs on other banks. Each individual bank essentially faces a run by its own creditors and "runs" by other banks via the asset market. Rollover decisions of creditors and asset market prices are jointly determined in equilibrium. Our model with this two-layer structure — creditors run on banks and banks with heterogeneous fundamentals interact in the asset market — offers insights that cannot be obtained from a model with a single-layer structure. Our model also has unique policy implications. The model framework is highly tractable and extendable and hence potentially useful for future research.

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Appendix

A Proofs

Proof of Lemma 1: Given that the price l_i fully reveals the fundamentals θ_i, an investor does not rely on his private information in trading. Hence, all investors are basically the same (as a representative investor). Thus, the objective function of (1) can be transformed into one maximizing $\int q_i(\theta_i - l_i)di - \frac{1}{2}\gamma Var\left(e\int q_i di\right) = \int q_i(\theta_i - l_i)di - \frac{1}{2}\gamma\sigma_e^2\left(\int q_i di\right)^2$.

The FOC with respect to any q_i implies $(\theta_i - l_i)di - \gamma\sigma_e^2\left(\int q_i di\right)di = 0$, that is, $\int q_i di = \frac{\theta_i - l_i}{\gamma\sigma_e^2}$ for any i. Different risky assets are perfect substitutes as long as their risk premium is the same. Because $n\int q_i di = \varphi$ by the market-clearing condition, we have $l_i = \theta_i - \varphi/k$, where $k \equiv n/\left(\gamma\sigma_e^2\right)$.

Proof of Lemmas 2 and 3: The distribution of v_i under a higher θ_i has first-order stochastic dominance over that under a lower θ_i. Because function $\min\left[R, \frac{v_i}{F}\right]$ is non-decreasing in v_i and strictly increasing for some ranges of v_i, $D(\theta_i; R) \equiv \mathbb{E}\left(\min\left[R, \frac{v_i}{F}\right]|\theta_i\right)$ is increasing in θ_i. Also, $\lim_{\theta_i \to +\infty} D(\theta_i; R) = R$.

By (4), we can obtain θ^* as a function of s^*, that is, function $\theta^*(s^*)$ is given by the implicit function $s^* = \theta^* + \sigma_s\Phi^{-1}\left(\frac{\theta^* - \varphi/k}{F}\right)$. Then, we replace θ^* in (5) with s^* by plugging in the function $\theta^*(s^*)$. Clearly, (5) can be rewritten as

$$\int_{\theta_i=\theta^*}^{+\infty} \left(D(\theta_i) - 1\right) d\Phi\left(\frac{\theta_i - \left(\frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta + \frac{\tau_s}{\tau_\theta+\tau_s}s^*\right)}{\sqrt{\frac{1}{\tau_\theta+\tau_s}}}\right) = \int_{\theta_i=-\infty}^{\theta_i=\theta^*} \Delta d\Phi\left(\frac{\theta_i - \left(\frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta + \frac{\tau_s}{\tau_\theta+\tau_s}s^*\right)}{\sqrt{\frac{1}{\tau_\theta+\tau_s}}}\right). \tag{A.1}$$

Write the LHS of (A.1) as $V^L(s^*; \sigma_s)$. We transform $V^L(s^*; \sigma_s)$ by changing variables to $z = \frac{\theta_i - \left(\frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta + \frac{\tau_s}{\tau_\theta+\tau_s}s^*\right)}{\sqrt{\frac{1}{\tau_\theta+\tau_s}}}$ and obtain $V^L(s^*; \sigma_s) = \int_{z=z_0}^{\infty}\left[D\left(\sqrt{\frac{1}{\tau_\theta+\tau_s}}z + \left(\frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta + \frac{\tau_s}{\tau_\theta+\tau_s}s^*\right)\right) - 1\right]\phi(z)\,dz$, where z_0 satisfies the joint equations

$$z_0 = \frac{\theta^* - \left(\frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta + \frac{\tau_s}{\tau_\theta+\tau_s}s^*\right)}{\sqrt{\frac{1}{\tau_\theta+\tau_s}}} \quad \text{and} \quad s^* = \theta^* + \sigma_s\Phi^{-1}\left(\frac{\theta^* - \varphi/k}{F}\right). \tag{A.2}$$

By (A.2), we have $s^* - \left[z_0\sqrt{\frac{1}{\tau_\theta+\tau_s}} + \left(\frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta + \frac{\tau_s}{\tau_\theta+\tau_s}s^*\right)\right] = \sigma_s\Phi^{-1}\left(\frac{\theta^* - \varphi/k}{F}\right)$, or

$$z_0 = \frac{\frac{\tau_\theta}{\tau_\theta+\tau_s}(s^* - \mu_\theta)}{\sqrt{\frac{1}{\tau_\theta+\tau_s}}} - \frac{\sigma_s}{\sqrt{\frac{1}{\tau_\theta+\tau_s}}}\Phi^{-1}\left(\frac{\left(\frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta + \frac{\tau_s}{\tau_\theta+\tau_s}s^*\right) + z_0\sqrt{\frac{1}{\tau_\theta+\tau_s}} - \varphi/k}{F}\right).$$

So it follows that $\lim_{\sigma_s \to 0} z_0 = -\Phi^{-1}\left(\frac{s^* - \varphi/k}{F}\right)$. Thus, under the limit $\sigma_s \to 0$ for a given τ_θ, we have $\theta^* = s^*$ and $\lim_{\sigma_s \to 0} V^L(s^*; \sigma_s) = (D(s^*) - 1)\cdot\int_{-\Phi^{-1}\left(\frac{s^* - \varphi/k}{F}\right)}^{\infty}\phi(z)\,dz = (D(s^*) - 1)\cdot\frac{s^* - \varphi/k}{F}$. Similarly, writing the RHS of (A.1) as $V^R(s^*; \sigma_s)$, we have $\lim_{\sigma_s \to 0} V^R(s^*; \sigma_s) = \Delta\cdot\left(1 - \frac{s^* - \varphi/k}{F}\right)$. Therefore,

(7) is proved. Clearly, (7) has a unique solution s^* and hence a unique equilibrium. Note that an equilibrium s^* must satisfy the conditions $0 < \frac{s^*-\varphi/k}{F} \leq 1$ and $D(s^*) \geq 1$.

Second, we prove that a creditor rolls over when his signal is higher than s^* and otherwise withdraws. Denote by s^{*h} the threshold used by an individual creditor and by s^* the threshold used by other creditors of the same bank as well as by creditors of other banks. Writing the LHS minus the RHS of (A.1) as $\tilde{V}\left(s^{*h}; s^*, \mu_\theta, \sigma_s\right)$, we obtain

$$\tilde{V}\left(s^{*h}; s^*, \mu_\theta, \sigma_s\right) = \left\{ \begin{array}{l} \int_{\theta_i=\theta^*}^{+\infty} \left(D(\theta_i) - 1\right) d\Phi\left(\frac{\theta_i - \left(\frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta + \frac{\tau_s}{\tau_\theta+\tau_s}s^{*h}\right)}{\sqrt{\frac{1}{\tau_\theta+\tau_s}}}\right) \\ -\int_{\theta_i=-\infty}^{\theta_i=\theta^*} \Delta d\Phi\left(\frac{\theta_i - \left(\frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta + \frac{\tau_s}{\tau_\theta+\tau_s}s^{*h}\right)}{\sqrt{\frac{1}{\tau_\theta+\tau_s}}}\right) \end{array} \right\}, \tag{A.3}$$

where $\theta^* = \theta^*(s^*)$ is given by the implicit function $s^* = \theta^* + \sigma_s \Phi^{-1}\left(\frac{\theta^*-\varphi/k}{F}\right)$, implying that θ^* is an increasing function of s^*. An individual creditor h takes s^* or θ^* as given. It is easy to obtain $\frac{\partial \tilde{V}}{\partial s^{*h}} > 0$ and $\frac{\partial \tilde{V}}{\partial s^*} < 0$ at $\left(s^{*h} = s^*,\, s^*\right)$ with $\tilde{V}\left(s^{*h}; s^*, \mu_\theta, \sigma_s\right) = 0$.

By changing variables to $z = \frac{\theta_i - \Xi}{\sqrt{\frac{1}{\tau_\theta+\tau_s}}}$ with $\Xi = \frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta + \frac{\tau_s}{\tau_\theta+\tau_s}s^{*h}$, we obtain $\tilde{V}\left(s^{*h}; s^*, \mu_\theta, \sigma_s\right) = \left\{ \int_{z=\frac{\theta^*-\Xi}{\sqrt{\frac{1}{\tau_\theta+\tau_s}}}}^{+\infty} \left[D\left(\sqrt{\frac{1}{\tau_\theta+\tau_s}}z + \Xi\right) - 1\right]\phi(z)\,dz - \int_{z=-\infty}^{\frac{\theta^*-\Xi}{\sqrt{\frac{1}{\tau_\theta+\tau_s}}}} \Delta \cdot \phi(z)\,dz \right\}$. Define $\hat{V}(s^*) := (D(s^*) - 1)\frac{s^*-\varphi/k}{F} - \Delta\left(1 - \frac{s^*-\varphi/k}{F}\right)$. Clearly, $\hat{V}(s^*) = \lim_{\sigma_s \to 0} \tilde{V}\left(s^{*h} = s^*; s^*, \mu_\theta, \sigma_s\right)$. A symmetric equilibrium under $\sigma_s \to 0$ is given by $\hat{V}(s^*) = 0$, which is equation (7). Note that function $V(s^*)$ on the LHS of (10) is a linear transformation of $\hat{V}(s^*)$, that is, $V(s^*) = \frac{1}{\Delta}\hat{V}(s^*) + 1$. Hence, equation $\hat{V}(s^*) = 0$ is equivalent to equation $V(s^*) = 1$.

Next, we prove the comparative statics. It is straightforward to show that $\frac{\partial \varphi}{\partial s^*} > 0$ in (9). As for $\frac{\partial s^*}{\partial \varphi} > 0$ in (7), (7) corresponds to equation $\hat{V}(s^*, \varphi) = 0$ and we have $\frac{\partial \hat{V}}{\partial s^*} = \frac{dD(s^*)}{ds^*}\frac{s^*-\varphi/k}{F} + \frac{1}{F}(D(s^*) - 1 + \Delta) > 0$ and $\frac{\partial \hat{V}}{\partial \varphi} = (D(s^*) - 1 + \Delta)\left(\frac{-1/k}{F}\right) < 0$, and thus $\frac{\partial s^*}{\partial \varphi} = -\frac{\partial \hat{V}}{\partial \varphi}/\frac{\partial \hat{V}}{\partial s^*} > 0$.

Finally, we show that a stable equilibrium corresponds to $\frac{d\hat{V}(s^*)}{ds^*} > 0$ at the equilibrium solution s^* to $\hat{V}(s^*) = 0$ and an unstable equilibrium corresponds to $\frac{d\hat{V}(s^*)}{ds^*} < 0$ at the equilibrium solution s^*. At an equilibrium point $\left(s^{*h} = s^*, s^*\right)$, it is a stable equilibrium if and only if $\frac{d\hat{V}(s^*)}{ds^*} > 0$ is satisfied. This is because

$$\begin{aligned} &\frac{d\hat{V}(s^*)}{ds^*} > 0 \Leftrightarrow \tilde{V}\left(s^{*h} + \Delta, s^* + \Delta\right) - \tilde{V}\left(s^{*h}, s^*\right) > 0 \text{ for a small } \Delta > 0 \\ \Leftrightarrow \quad &\frac{\partial \tilde{V}}{\partial s^{*h}} + \frac{\partial \tilde{V}}{\partial s^*} > 0 \Leftrightarrow \frac{\partial s^{*h}}{\partial s^*} = -\frac{\partial V/\partial s^*}{\partial \tilde{V}/\partial s^{*h}} < 1, \end{aligned}$$

by noting that $\frac{\partial \tilde{V}}{\partial s^{*h}} > 0$ and $\frac{\partial \tilde{V}}{\partial s^*} < 0$. Similarly, at an equilibrium point $\left(s^{*h} = s^*, s^*\right)$, it is an unstable equilibrium if and only if $\frac{dV(s^*)}{ds^*} < 0$ is satisfied. As $V(s^*)$ is a linear transformation of $\hat{V}(s^*)$, a stable equilibrium also corresponds to $\frac{\partial V(s^*)}{\partial s^*} > 0$ at the equilibrium point s^* and an unstable equilibrium to $\frac{\partial V(s^*)}{\partial s^*} < 0$ at the equilibrium point s^*.

Proof of Proposition 1: Write the LHS of (10) as function $V(s^*;\mu_\theta,k)$, where

$$V(s^*;\mu_\theta,k)=\left\{\frac{1}{F}\left[s^*-\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)\Big/k\right]\right\}\frac{D(s^*)-1+\Delta}{\Delta}. \tag{A.4}$$

It is easy to show that

$$\begin{aligned}\frac{\partial V(s^*;\mu_\theta,k)}{\partial s^*}&=\frac{1}{F}\left[1-\phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)\frac{1}{k\sigma_\theta}\right]\frac{D(s^*)-1+\Delta}{\Delta}+\frac{1}{F}\left[s^*-\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)\Big/k\right]\frac{D'(s^*)}{\Delta},\\ \frac{\partial V(s^*;\mu_\theta,k)}{\partial \mu_\theta}&=\left\{\frac{1}{F}\left[\frac{1}{\sigma_\theta}\phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)\Big/k\right]\right\}\frac{D(s^*)-1+\Delta}{\Delta}>0,\\ \frac{\partial V(s^*;\mu_\theta,k)}{\partial k}&=\left\{\frac{1}{F}\left[\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)\Big/k^2\right]\right\}\frac{D(s^*)-1+\Delta}{\Delta}>0.\end{aligned} \tag{A.5}$$

By (A.5), when s^* is sufficiently higher or lower than μ_θ, $1-\phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)>0$ and hence $\frac{\partial V(s^*;\mu_\theta,k)}{\partial s^*}>0$. When s^* is at an intermediate level close to μ_θ, it can be $\frac{\partial V(s^*;\mu_\theta,k)}{\partial s^*}<0$ and thus $V(s^*;\mu_\theta,k)$ can be non-monotonic in s^*. Because there can be at most only one continuous interval around μ_θ in which $\frac{\partial V(s^*;\mu_\theta,k)}{\partial s^*}<0$, the non-monotonic curve of $V(s^*;\mu_\theta,k)$ is "N"-shaped in s^* (i.e., increasing first and then decreasing before increasing again). So $V(s^*;\mu_\theta,k)=1$ can admit one or (generically) three solutions with respect to s^*.

i) When k is high enough, $\frac{\partial V(s^*;\mu_\theta,k)}{\partial s^*}>0$ also holds for any s^*, so $V(s^*;\mu_\theta,k)=1$ admits a unique solution with respect to s^*.

When μ_θ decreases, the curve $V(s^*;\mu_\theta,k)$ shifts downward in Figure 5. When μ_θ is low enough, the curve $V(s^*;\mu_\theta,k)$ intersects the horizontal line $V=1$ only once for any k, so $V(s^*;\mu_\theta,k)=1$ admits a unique solution with respect to s^*.

ii) Consider a sufficiently high μ_θ such that the curve $V(s^*;\mu_\theta,k)$ could intersect the horizontal line $V=1$ more than once for some k. A decrease in k not only increases the curvature of $V(s^*;\mu_\theta,k)$ but also shifts the curve $V(s^*;\mu_\theta,k)$ downward by $\frac{\partial V(s^*;\mu_\theta,k)}{\partial k}>0$. Hence, when k is sufficiently high or sufficiently low, $V(s^*;\mu_\theta,k)=1$ admits a unique solution with respect to s^*. When k is not too high and not too low, $V(s^*;\mu_\theta,k)=1$ admits multiple (typically three) solutions with respect to s^*.

iii) Consider a sufficiently low k such that $\frac{\partial V(s^*;\mu_\theta,k)}{\partial s^*}<0$ at some s^* (i.e., $V(s^*;\mu_\theta,k)$ is non-monotonic with respect to s^*). Considering that the curve $V(s^*;\mu_\theta,k)$ shifts downward as μ_θ decreases, $V(s^*;\mu_\theta,k)=1$ admits multiple (typically three) solutions with respect to s^* when μ_θ is not too high and not too low and a unique solution when μ_θ is sufficiently high or sufficiently low.

iv) Based on the result in the proof of Lemma 2, at a stable equilibrium $\frac{\partial V(s^*)}{\partial s^*}>0$. Because $\frac{\partial V(s^*;\mu_\theta,k)}{\partial \mu_\theta}>0$ and $\frac{\partial V(s^*;\mu_\theta,k)}{\partial k}>0$, it follows that $\frac{\partial s^*}{\partial \mu_\theta}=-\frac{\partial V/\partial \mu_\theta}{\partial V/\partial s^*}<0$ and $\frac{\partial s^*}{\partial k}=-\frac{\partial V/\partial k}{\partial V/\partial s^*}<0$. Considering $\varphi=\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$, it follows that $\frac{\partial \varphi}{\partial \mu_\theta}<0$ and $\frac{\partial \varphi}{\partial k}<0$.

Proof of Lemma 4: The proof is straightforward and hence omitted.

Proof of Lemma 5: Plugging (14) into (15), it follows that

$$\frac{s^{**}-(\eta\varphi)/k}{s^{**}}\int_{\theta_i=-\infty}^{s^{**}}(1-c)\,\theta_i\cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right)=\int_{\theta_i=s^*}^{+\infty}c\cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right), \tag{A.6}$$

where $\eta\varphi = (1-c)\left(\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right) - \Phi\left(\frac{s^{**}-\mu_\theta}{\sigma_\theta}\right)\right)$. When $c = 0$, the equilibrium reverts to the baseline model. Consider $c > 0$. For a given (s^*, c), the LHS of (A.6) is increasing in s^{**}. Therefore, when c is sufficiently high such that $c\left[1 - \Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)\right] \geq (1-c)\left[\int_{\theta_i=-\infty}^{s^*} \theta_i \cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right)\right]$, we have the corner solution $s^{**} = s^*$, and thus $\eta = 0$, $l_i = \theta_i$ for all i, and $I = 1$; otherwise there is a unique solution $s^{**} \in (\underline{s}, s^*)$, where the lower bound $\underline{s}$ is the solution to $I|_{s^{**}=\underline{s}} = +\infty$ or $\underline{s} - (1-c)\left(\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right) - \Phi\left(\frac{\underline{s}-\mu_\theta}{\sigma_\theta}\right)\right)/k = 0$. After obtaining s^{**}, we can find $l(\cdot)$, I, $\hat{l}(\cdot)$, and $\tilde{I}$.

Proof of Proposition 2: The equilibrium solves the system of equations (A.6) and (18).

1) The effect of k. First, we show that (A.6) gives $\frac{\partial(\eta\varphi)}{\partial s^*} > 0$, $\frac{\partial I}{\partial s^*} > 0$, $\frac{\partial((\eta\varphi)/k)}{\partial k} < 0$, and $\frac{\partial I}{\partial k} < 0$. For $\frac{\partial(\eta\varphi)}{\partial s^*} > 0$, we prove by contradiction. Suppose an increase in s^* leads to $\eta\varphi$ decreasing, which implies that s^{**} must be increasing by $\eta\varphi = (1-c)\left(\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right) - \Phi\left(\frac{s^{**}-\mu_\theta}{\sigma_\theta}\right)\right)$; then the LHS of (A.6) is increasing in s^*. This forms a contradiction because the RHS of (A.6) is decreasing in s^*. After obtaining $\frac{\partial(\eta\varphi)}{\partial s^*} > 0$, we prove that (A.6) also gives $\frac{\partial I}{\partial s^*} > 0$. Suppose an increase in s^* leads to I decreasing, which implies that s^{**} must be decreasing to maintain equality in (15). This forms a contradiction to $\frac{1}{I} = \frac{s^{**}-(\eta\varphi)/k}{s^{**}}$ by considering that we have proved $\frac{\partial(\eta\varphi)}{\partial s^*} > 0$. We turn to proving $\frac{\partial((\eta\varphi)/k)}{\partial k} < 0$ by contradiction. Suppose an increase in k leads to $(\eta\varphi)/k$ increasing, which implies that $\eta\varphi$ must be increasing, which then implies that s^{**} is decreasing by $\eta\varphi = (1-c)\left(\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right) - \Phi\left(\frac{s^{**}-\mu_\theta}{\sigma_\theta}\right)\right)$; then, on the LHS of (A.6), both $\frac{s^{**}-(\eta\varphi)/k}{s^{**}}$ and s^{**} are decreasing, so equality in (A.6) cannot be true, which forms a contradiction. After obtaining $\frac{\partial((\eta\varphi)/k)}{\partial k} > 0$, we also have $\frac{\partial I}{\partial k} < 0$, with a similar logic as earlier. Second, it is easy to show that (18) gives $\frac{\partial s^*}{\partial((\eta\varphi)/k)} > 0$, similar to the proof for $\frac{\partial s^*}{\partial \varphi} > 0$ in (7). Also, a change in $\tilde{I}$ caused by a change in s^{**} affects $\hat{D}(s^*)$, but the effect is small under a sufficient condition that σ_e is big enough. Third, combining the two steps above yields the feedback loop: $\frac{\partial((\eta\varphi)/k)}{\partial k} < 0$ and $\frac{\partial((\eta\varphi)/k)}{\partial s^*} > 0$ in (A.6) and $\frac{\partial s^*}{\partial((\eta\varphi)/k)} > 0$ in (18). The loop is similar to the one in Lemma 3 and Proposition 1. Overall, $\frac{\partial s^*}{\partial k} < 0$, $\frac{\partial((\eta\varphi)/k)}{\partial k} < 0$, and $\frac{\partial I}{\partial k} < 0$.

2) The effect of c. (A.6) gives $\frac{\partial(\eta\varphi)}{\partial c} < 0$ and $\frac{\partial I}{\partial c} < 0$, which can be proved by contradiction as in 1). Write the LHS of (18) as $V(s^*; c) = \frac{c+(1-c)[s^*-(\eta\varphi)/k]}{F} \cdot \frac{\hat{D}(s^*)-1+\Delta}{\Delta}$, implying $\frac{\partial V(s^*;c)}{\partial c} = \frac{[1-(s^*-(\eta\varphi)/k)]}{F}\frac{\hat{D}(s^*;c)-1+\Delta}{\Delta} + \frac{c+(1-c)\cdot(s^*-(\eta\varphi)/k)}{F}\frac{1}{\Delta}\frac{\partial \hat{D}(s^*;c)}{\partial c}$. Note that $s^* - (\eta\varphi)/k < 1$ by considering that $\frac{c+(1-c)\cdot(s^*-(\eta\varphi)/k)}{F} \leq 1$; moreover, $\frac{\partial \hat{D}(s^*;c)}{\partial c}$ is positive or a small negative number by noting that it can be $I > s^*$ when c is not large. Overall, $\frac{\partial V(s^*;c)}{\partial c} > 0$ when c is not large. Hence, by the implicit function theorem, $\frac{\partial s^*}{\partial c} = -\frac{\partial V(s^*;c)}{\partial c}/\frac{\partial V(s^*;c)}{\partial s^*} < 0$, noting that at a stable equilibrium $\frac{\partial V(s^*;c)}{\partial s^*} > 0$ (see the proof of Lemma 2). So (18) gives $\frac{\partial s^*}{\partial((\eta\varphi)/k)} > 0$ and $\frac{\partial s^*}{\partial c} < 0$. Thus, (A.6) and (18) together form a feedback loop: $\frac{\partial((\eta\varphi)/k)}{\partial c} < 0$ and $\frac{\partial((\eta\varphi)/k)}{\partial s^*} > 0$ in (A.6) and $\frac{\partial s^*}{\partial((\eta\varphi)/k)} > 0$ and $\frac{\partial s^*}{\partial c} < 0$ in (18). Hence, $\frac{\partial s^*}{\partial c} < 0$, $\frac{\partial((\eta\varphi)/k)}{\partial c} < 0$, and $\frac{\partial I}{\partial c} < 0$. A change in c has a direct effect on s^* through $\frac{\partial s^*}{\partial c} < 0$ by (18) and also an indirect effect on s^* through the combination of $\frac{\partial((\eta\varphi)/k)}{\partial c} < 0$ by (A.6) and $\frac{\partial s^*}{\partial((\eta\varphi)/k)} > 0$ by (18), that is, $\frac{ds^*}{dc} < \frac{\partial s^*}{\partial c} < 0$, where $\frac{ds^*}{dc}$ denotes the total effect.

Write the LHS of (18) as $V(s^*; c, k) = \frac{c+(1-c)[s^*-(\eta\varphi)/k]}{F} \cdot \frac{\hat{D}(s^*)-1+\Delta}{\Delta}$, where $\eta\varphi$ and $\tilde{I}$ in $\hat{D}(s^*)$ are endogenous and given by (A.6) and (17). Figure A1 plots equation $V(s^*; c, k) = 1$ under a set of parameter values $\mu_\theta = 1.6$, $\sigma_\theta = 0.6$, $F = 0.6$, $R = 1.1$, $\sigma_e = 0.2$, and $\Delta = 0.6$. Under the same set of parameter values, Figure A2 plots comparative statics in Proposition 2, where $k = 0.28$ in the left panel and $c = 0.003$ in the right panel.

Liquidity holdings of banks affect systemic runs through three channels. First, banks rely less on the asset market to fetch liquidity to accommodate creditors' early withdrawals and hence expectations about the aggregate market condition become less important. Second, the liquidity holdings of peer (strong) banks enter the asset market and provide a cushion for the downward-sloping fire-sale prices determined by risk-averse outside investors. The market depth k is in effect increased. The effective market depth can be expressed as $\hat{k} \equiv k/\eta > k$ based on (13). Third, the asset market condition can be more sensitive to expectations. Specifically, a higher s^* corresponds to more demand for liquidity coupled with less supply — two joint forces (see (15)), in contrast to the baseline model in which a higher s^* corresponds to more demand (φ) but no change on the supply side. The forces of the first two channels make expectation-driven equilibrium multiplicity less likely whereas the force of the third channel makes it more likely. We see in Figure A1 that the curvature of $V(s^*; c, k)$ can be higher under higher cash c than under lower c in some regions of s^*.

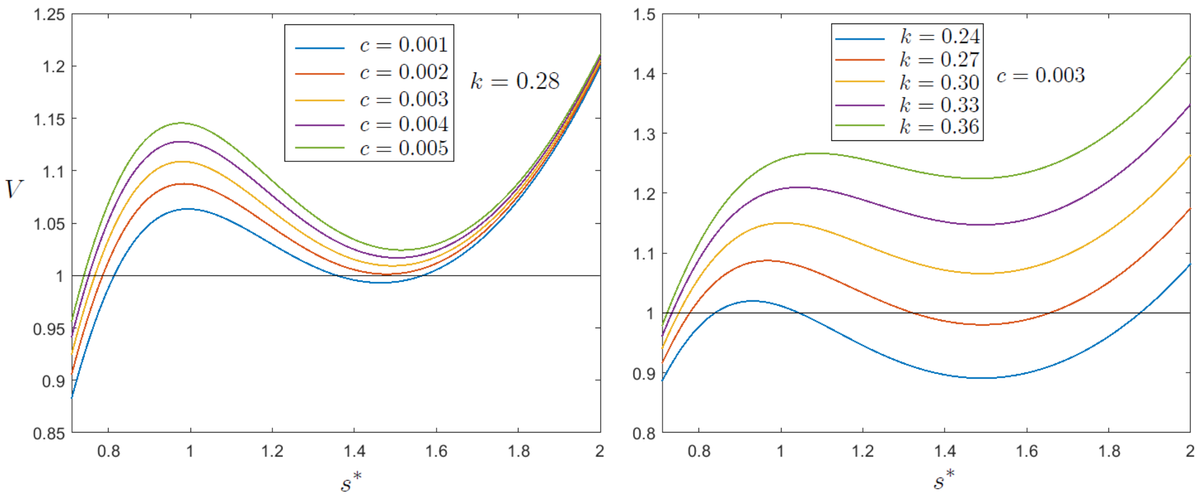

Figure A1. Equation $V(s^*; c, k) = 1$.

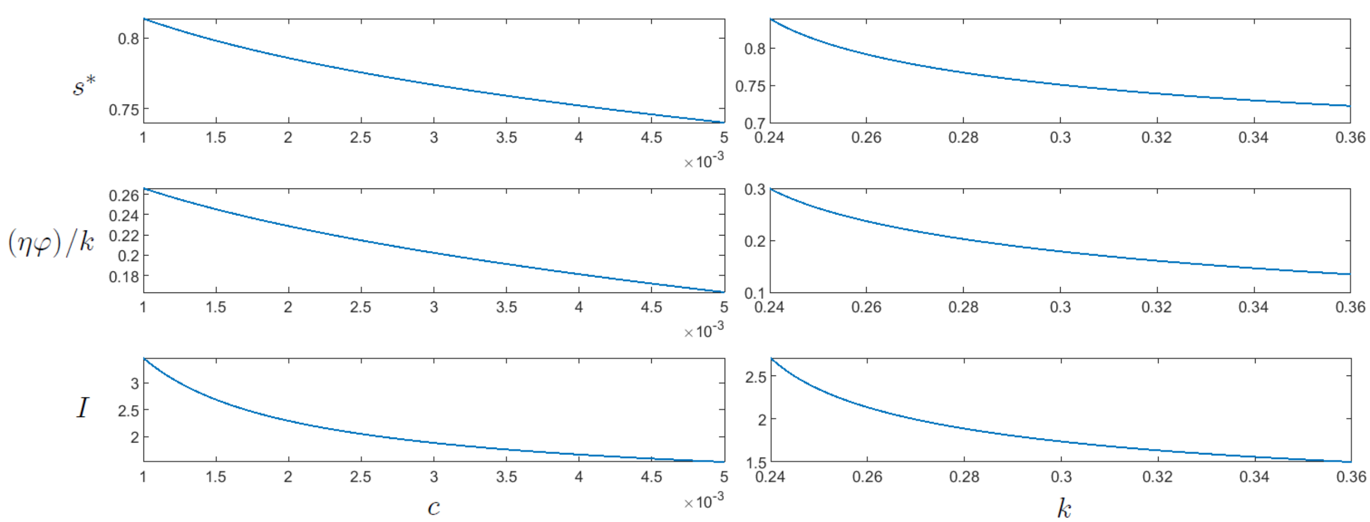

Figure A2. Comparative statics in Proposition 2.

Proof of Proposition 3: Due to the complexity of the general-equilibrium nature of the model with many endogenous variables, we focus on characterizing the economic forces behind the comparative statics, show that the result holds under some conditions, and confirm the result by the simulation exercise for a large parameter space. The logic of the proof is as follows. In the first step, suppose the interbank market is closed (i.e., banks that realize stronger fundamentals and thus face fewer interim withdrawals hold their liquidity, rather than supplying liquidity to the asset market) and calculate the net amount of excess liquidity Π. In the second step, let the interbank market open and examine how the equilibrium changes to clear the asset (interbank) market, and evaluate the effect on $\eta\varphi$.

We start from the first step. Consider the special case of a simple binomial distribution of $G(\cdot)$ to illustrate the intuition. Specifically, assume $c_i \in \left\{c^A,\ c^B\right\}$ with equal probability for each realization, that is, half of the banks have a higher amount c^A of liquidity (type-A banks) while the other half have a lower amount c^B (type-B banks), where $c^A > c^B$. Then (24) becomes

$$\Pi\left(c^A,\, c^B\right) = \left\{ \begin{array}{l} \frac{1}{2}\left[c^A \int_{\theta_i = s^*(c^A)}^{+\infty} d\Phi\left(\frac{\theta_i - \mu_\theta}{\sigma_\theta}\right) + c^B \int_{\theta_i = s^*(c^B)}^{+\infty} d\Phi\left(\frac{\theta_i - \mu_\theta}{\sigma_\theta}\right)\right] \\ -\frac{1}{2}\left[\int_{\theta_i = -\infty}^{s^*(c^A)} \theta_i \cdot d\Phi\left(\frac{\theta_i - \mu_\theta}{\sigma_\theta}\right) + \int_{\theta_i = -\infty}^{s^*(c^B)} \theta_i \cdot d\Phi\left(\frac{\theta_i - \mu_\theta}{\sigma_\theta}\right)\right] \end{array} \right\}.$$

Now consider a mean-preserving spread: $c_i \in \left\{c^A + \Delta_c,\ c^B - \Delta_c\right\}$, where $\Delta_c > 0$. When $\Delta_c \to 0$,

$$\begin{aligned} &\Pi\left(c^A + \Delta_c,\, c^B - \Delta_c\right) - \Pi\left(c^A,\, c^B\right) \\ = &\left\{ \underbrace{\frac{1}{2}\Delta_c\ \left(\Pr\left(\theta_i > s^*\left(c^A\right)\right) - \Pr\left(\theta_i > s^*\left(c^B\right)\right)\right)}_{\text{additional amount } \Delta_c \text{ more likely remain in banking system}} + \frac{1}{2}\left[\left(c^A + s^*\left(c^A\right)\right)\Delta_{pA} - \left(c^B + s^*\left(c^B\right)\right)\Delta_{pB}\right] \right\}, \end{aligned} \tag{A.7}$$

where $\Delta_{pA} = \left(-\frac{\partial s^*(c)}{\partial c}\right)\frac{1}{\sigma_\theta}\phi\left(\frac{s^*(c) - \mu_\theta}{\sigma_\theta}\right)\Big|_{c = c^A} \cdot \Delta_c$ and $\Delta_{pB} = \left(-\frac{\partial s^*(c)}{\partial c}\right)\frac{1}{\sigma_\theta}\phi\left(\frac{s^*(c) - \mu_\theta}{\sigma_\theta}\right)\Big|_{c = c^B} \cdot \Delta_c$. Under the sufficient condition that k is high enough, the indirect effect of $\{c_i\}$ through φ on s^* is small relative to the direct effect of c_i on s^* in (22). The first term in the second line of (A.7) is positive by $s^*\left(c^A\right) < s^*\left(c^B\right)$. The second term is also positive under certain conditions. First, when (μ_θ, k) is in the region such that the crisis is severe with $\mu_\theta < s^*\left(c^A\right) < s^*\left(c^B\right)$, it follows that $\phi\left(\frac{s^*(c^A) - \mu_\theta}{\sigma_\theta}\right) > \phi\left(\frac{s^*(c^B) - \mu_\theta}{\sigma_\theta}\right)$ and consequently $\Delta_{pA} > \Delta_{pB}$ when σ_θ is small enough, by noting that the force of $-\frac{\partial s^*(c)}{\partial c}$ is dominated by the force of $\phi\left(\frac{s^*(c) - \mu_\theta}{\sigma_\theta}\right)$ when σ_θ is small enough. Second, $c^A > c^B$ while $s^*\left(c^A\right) < s^*\left(c^B\right)$, which implies that the difference between $c^A + s^*\left(c^A\right)$ and $c^B + s^*\left(c^B\right)$ can be small, and the force $\Delta_{pA} > \Delta_{pB}$ dominates when σ_θ is small enough, so $\left(c^A + s^*\left(c^A\right)\right)\Delta_{pA} - \left(c^B + s^*\left(c^B\right)\right)\Delta_{pB} > 0$.

The economic intuition is as follows. When the crisis is severe (i.e., $\mu_\theta < s^*\left(c^A\right) < s^*\left(c^B\right)$), it is optimal to give an additional amount of liquidity to relatively cash-rich banks rather than to relatively cash-poor banks. The additional amount of liquidity can significantly increase the survival probability for the latter type of banks while increasing little that for the former type (i.e., $\Delta_{pA} > \Delta_{pB}$). Moreover, the additional amount is more likely to stay in the banking system (rather than end in depositors' pockets) because the relatively cash-rich banks have a higher probability of surviving in the first place.

Consider the general case. Denote $\Lambda_S(c_i) = c_i\left[1-\Phi\left(\frac{s^*(c_i)-\mu_\theta}{\sigma_\theta}\right)\right]$ and $\Lambda_D(c_i) = \Gamma(s^*(c_i)) = \int_{\theta_i=-\infty}^{s^*(c_i)} \theta_i d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right)$, and $\Lambda(c_i) = \Lambda_S(c_i) - \Lambda_D(c_i)$. Hence,

$$\begin{aligned}
\frac{\partial \Lambda_S(c_i)}{\partial c_i} &= \left[1-\Phi\left(\frac{s^*(c_i)-\mu_\theta}{\sigma_\theta}\right)\right] + c_i\left[\phi\left(\frac{s^*(c_i)-\mu_\theta}{\sigma_\theta}\right)\frac{1}{\sigma_\theta}\left(-\frac{ds^*(c_i)}{dc_i}\right)\right] \\
\frac{\partial \Lambda_D(c_i)}{\partial c_i} &= s^*(c_i)\left[\phi\left(\frac{s^*(c_i)-\mu_\theta}{\sigma_\theta}\right)\frac{1}{\sigma_\theta}\left(\frac{ds^*(c_i)}{dc_i}\right)\right] \\
\frac{\partial \Lambda(c_i)}{\partial c_i} &= \left[1-\Phi\left(\frac{s^*(c_i)-\mu_\theta}{\sigma_\theta}\right)\right] + (c_i + s^*(c_i))\left[\phi\left(\frac{s^*(c_i)-\mu_\theta}{\sigma_\theta}\right)\frac{1}{\sigma_\theta}\left(-\frac{ds^*(c_i)}{dc_i}\right)\right].
\end{aligned}$$

To show $\frac{d\Lambda^2(c_i)}{dc_i^2} > 0$ is to show $\left.\frac{\partial\Lambda(c_i)}{\partial c_i}\right|_{c=c^A} > \left.\frac{\partial\Lambda(c_i)}{\partial c_i}\right|_{c=c^B}$ for $c^A > c^B$. It follows that

$$\begin{aligned}
\left.\frac{\partial\Lambda(c_i)}{\partial c_i}\right|_{c_i=c^A} - \left.\frac{\partial\Lambda(c_i)}{\partial c_i}\right|_{c_i=c^B} &= \left[\Phi\left(\frac{s^*(c^B)-\mu_\theta}{\sigma_\theta}\right) - \Phi\left(\frac{s^*(c^A)-\mu_\theta}{\sigma_\theta}\right)\right] \\
&+ \left(\begin{array}{c} \left.\left\{(c_i+s^*(c_i))\left[\phi\left(\frac{s^*(c_i)-\mu_\theta}{\sigma_\theta}\right)\frac{1}{\sigma_\theta}\left(-\frac{ds^*(c_i)}{dc_i}\right)\right]\right\}\right|_{c_i=c^A} \\ -\left.\left\{(c_i+s^*(c_i))\left[\phi\left(\frac{s^*(c_i)-\mu_\theta}{\sigma_\theta}\right)\frac{1}{\sigma_\theta}\left(-\frac{ds^*(c_i)}{dc_i}\right)\right]\right\}\right|_{c_i=c^B} \end{array}\right),
\end{aligned}$$

where the first term is positive, and the second term is positive if $\mu_\theta < s^*(c^A) < s^*(c^B)$ and σ_θ is small enough. Similarly, $\frac{\partial\Lambda_S^2(c_i)}{\partial c_i^2} > 0$ when σ_θ is sufficiently small. Consider two distributions $G'(\cdot)$ and $G''(\cdot)$, where G' second-order stochastically dominates G''. Then, S^L and $\Pi = S^L - D^L$ are both higher under G'' than under G', while D^L can be higher or lower under G'' than under G'.

We proceed to the second step. When c is small enough, s^{**} is below $s^*(c_i)$ for all c_i. The market-clearing condition (21) can be rewritten as $\frac{D^L - D_M^L}{I} = S^L$, where $I = \frac{s^{**}}{s^{**}-(\eta\varphi)/k}$ with $\eta\varphi = \int\left[\Phi\left(\frac{s^*(c_i)-\mu_\theta}{\sigma_\theta}\right) - \Phi\left(\frac{s^{**}-\mu_\theta}{\sigma_\theta}\right)\right]dG(c_i)$, and $D_M^L = \int\left[\int_{\theta_i=s^{**}}^{s^*(c_i)}\theta_i d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right)\right]dG(c_i)$, meaning the aggregate value of assets that are absorbed by outside investors. For given $\{s^*(c_i)\}$, both D_M^L and I are decreasing in s^{**}. Thus, the change in the distribution of c_i from G' to G'', leading to S^L increasing more than D^L, requires an increase in s^{**} (and hence a decrease in D_M^L, I, and $\eta\varphi$) to maintain equality in $\frac{D^L - D_M^L}{I} = S^L$ when regularity is guaranteed under some parameter values, considering that $I > 1$. The simulation exercise confirms that the result in Proposition 3 holds true for a large parameter space. Figure 7 is an example.

Proof of Lemma 6: To solve the equilibrium analytically, we first obtain $\mathbb{E}_{\mu_\theta|\theta_i,s_i^h}\left(\cdot|\,\theta_i, s_i^h\right) = \mathbb{E}_{\mu_\theta|\theta_i}\left(\cdot|\,\theta_i\right)$ and then, by applying it to $\mathbb{E}_{\mu_\theta,\theta_i|s_i^h}\left(\cdot|\,s_i^h\right) = \mathbb{E}_{\theta_i|s_i^h}\left(\mathbb{E}_{\mu_\theta|\theta_i,s_i^h}\left(\cdot|\,\theta_i, s_i^h\right)\big|\, s_i^h\right)$ (the law of iterated expectations), have the following result: $\mathbb{E}_{\mu_\theta,\theta_i|s_i^h}\left(\cdot|\,s_i^h\right) = \mathbb{E}_{\theta_i|s_i^h}\left(\mathbb{E}_{\mu_\theta|\theta_i}\left(\cdot|\,\theta_i\right)\big|\, s_i^h\right)$, with which to calculate (27).

Recall the information (signal) structure: $\mu_\theta \sim N(\bar{\mu}_\theta, \sigma_{\mu_\theta}^2 = \tau_{\mu_\theta}^{-1})$, $\theta_i = \mu_\theta + \sigma_\theta\delta_i$ with $\delta_i \sim N(0,1)$, and $s_i^h = \theta_i + \sigma_s\epsilon_i^h$ with $\epsilon_i^h \sim N(0,1)$. The information structure is a hierarchical one: $\mu_\theta \to \theta_i \to s_i^h$, that is, θ_i is a signal about μ_θ and s_i^h is a signal about θ_i. We prove the result

$$\mathbb{E}_{\mu_\theta,\theta_i|s_i^h}\left(\cdot|\,s_i^h\right) = \mathbb{E}_{\theta_i|s_i^h}\left(\mathbb{E}_{\mu_\theta|\theta_i}\left(\cdot|\,\theta_i\right)\big|\, s_i^h\right), \tag{A.8}$$

where "$\cdot$" denotes a function with two variables θ_i and μ_θ. The proof has two steps. To simplify notation, consider a general case of the hierarchical information structure $x \to y \to s$, where y is a signal about x and s is a signal about y. First, it is easy to prove that $\mathbb{E}_{x|y,s}(\cdot|\, y, s) = \mathbb{E}_{x|y}(\cdot|\, y)$, that is, in the presence of signal y, signal s is redundant in inferring x. Intuitively, information $\{y\}$ is a sufficient statistic for $\{y, s\}$ in inferring x. Second, by the law of iterated expectations, $\mathbb{E}_{x,y|s}(\cdot|\, s) = \mathbb{E}_{y|s}\left(\mathbb{E}_{x|y,s}(\cdot|\, y, s)\middle|\, s\right)$, which, by substituting the result $\mathbb{E}_{x|y,s}(\cdot|\, y, s) = \mathbb{E}_{x|y}(\cdot|\, y)$ from the first step, yields $\mathbb{E}_{x,y|s}(\cdot|\, s) = \mathbb{E}_{y|s}\left(\mathbb{E}_{x|y}(\cdot|\, y)\middle|\, s\right)$. Hence, (A.8) is obtained.

Applying the law of iterated expectations to (27) with using the result in (A.8) yields

$$\mathbb{E}_{\theta_i|s_i^h}\left[\mathbb{E}_{\theta^*|\theta_i}\left((D(\theta_i)-1)\cdot \mathbf{1}_{\theta_i \geq \theta^*}\middle|\,\theta_i\right)\middle|\, s_i^h = s^{*h}\right] = \mathbb{E}_{\theta_i|s_i^h}\left[\mathbb{E}_{\theta^*|\theta_i}\left(\Delta\cdot \mathbf{1}_{\theta_i < \theta^*}\middle|\,\theta_i\right)\middle|\, s_i^h = s^{*h}\right],$$

where $\theta^* = \theta^*(s^*, \mu_\theta)$ is given by $\theta^* = \theta^*(s^*; \varphi(s^*, \mu_\theta))$ defined after (26). This implies that

$$\mathbb{E}_{\theta_i|s_i^h}\left[\left((D(\theta_i)-1)\cdot \Pr(\theta_i \geq \theta^*|\,\theta_i)\right)\middle|\, s_i^h = s^{*h}\right] = \mathbb{E}_{\theta_i|s_i^h}\left[\left(\Delta\cdot \Pr(\theta_i < \theta^*|\,\theta_i)\right)\middle|\, s_i^h = s^{*h}\right],$$

that is,

$$\begin{aligned}&\textstyle\int_{\theta_i=-\infty}^{+\infty}\left((D(\theta_i)-1)\cdot \Pr\left(\theta_i \geq \theta^*(s^*, \mu_\theta)|\,\theta_i\right)\right)\cdot f\left(\theta_i|s^{*h}\right)d\theta_i\\ &= \textstyle\int_{\theta_i=-\infty}^{+\infty}\left(\Delta\cdot \Pr\left(\theta_i < \theta^*(s^*, \mu_\theta)|\,\theta_i\right)\right)\cdot f\left(\theta_i|s^{*h}\right)d\theta_i,\end{aligned} \tag{A.9}$$

where $f\left(\theta_i|s^{*h}\right) = \sqrt{\Gamma}\phi\left(\frac{\theta_i - \left((\frac{1}{\tau_{\mu_\theta}}+\frac{1}{\tau_\theta})^{-1}\frac{1}{\Gamma}\bar{\mu}_\theta + \tau_s\frac{1}{\Gamma}s^{*h}\right)}{1/\sqrt{\Gamma}}\right)$ with $\Gamma \equiv 1/(\frac{1}{\tau_{\mu_\theta}}+\frac{1}{\tau_\theta}) + \tau_s$, by considering $\theta_i|s^{*h} \sim N\left((\frac{1}{\tau_{\mu_\theta}}+\frac{1}{\tau_\theta})^{-1}\frac{1}{\Gamma}\bar{\mu}_\theta + \tau_s\frac{1}{\Gamma}s^{*h}, \frac{1}{\Gamma}\right)$.

Plugging in $s^{*h} = s^*$, by changing variables to $z = \frac{\theta_i - \left((\frac{1}{\tau_{\mu_\theta}}+\frac{1}{\tau_\theta})^{-1}\frac{1}{\Gamma}\bar{\mu}_\theta + \tau_s\frac{1}{\Gamma}s^{*}\right)}{1/\sqrt{\Gamma}}$, the LHS of (A.9), denoted by $\digamma(s^*; \sigma_s)$, becomes

$$\digamma(s^*; \sigma_s) = \int_{z=-\infty}^{+\infty}\left(\left(D\left(z/\sqrt{\Gamma} + \left((\frac{1}{\tau_{\mu_\theta}}+\frac{1}{\tau_\theta})^{-1}\frac{1}{\Gamma}\bar{\mu}_\theta + \tau_s\frac{1}{\Gamma}s^*\right)\right)-1\right)\cdot \Pr(z \geq z^*|\, z)\right)\phi(z)\,dz,$$

where $z^* = \frac{\theta^*(s^*,\mu_\theta) - \left((\frac{1}{\tau_{\mu_\theta}}+\frac{1}{\tau_\theta})^{-1}\frac{1}{\Gamma}\bar{\mu}_\theta + \tau_s\frac{1}{\Gamma}s^{*}\right)}{1/\sqrt{\Gamma}}$. Because $\lim\limits_{\sigma_s\to 0} z^* = -\Phi^{-1}\left(\frac{s^*-\varphi/k}{F}\right)$ with $\varphi = \Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$ (the proof is similar to that of Lemma 2), we have $\lim\limits_{\sigma_s\to 0} z^* \in \left(-\Phi^{-1}\left(\frac{s^*}{F}\right), -\Phi^{-1}\left(\frac{s^*-1/k}{F}\right)\right)$ and

$$\begin{aligned}&\lim_{\sigma_s\to 0}\Pr(z \geq z^*|\, z) = \Pr\left(z \geq -\Phi^{-1}\left(\frac{s^* - \Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)/k}{F}\right)\middle|\, z\right)\\ &= \begin{cases} 1 & \text{for } z \in \left[-\Phi^{-1}\left(\frac{s^*-1/k}{F}\right), +\infty\right) \\ \Pr\left(\mu_\theta \geq s^* - \sigma_\theta\Phi^{-1}\left(k\left(s^* - F\Phi(-z)\right)\right)\middle|\, z\right) & \text{for } z \in \left(-\Phi^{-1}\left(\frac{s^*}{F}\right), -\Phi^{-1}\left(\frac{s^*-1/k}{F}\right)\right) \\ 0 & \text{for } z \in \left(-\infty, -\Phi^{-1}\left(\frac{s^*}{F}\right)\right] \end{cases}.\end{aligned}$$

As $\lim\limits_{\sigma_s\to 0}\Pr(z \geq z^*|\, z)$ is continuous and increasing in z, for simplicity we write $\lim\limits_{\sigma_s\to 0}\Pr(z \geq z^*|\, z) =$

$\Pr\left(\mu_\theta \geq s^* - \sigma_\theta \Phi^{-1}\left(k\left(s^* - F\Phi\left(-z\right)\right)\right) \middle| z\right)$. Then, taking the limit $\sigma_s \to 0$ on $\digamma\left(s^*; \sigma_s\right)$ yields

$$\begin{aligned}
\lim_{\sigma_s \to 0} \digamma\left(s^*; \sigma_s\right) &= \left(D\left(s^*\right) - 1\right) \int_{z=-\infty}^{+\infty} \Pr\left(\mu_\theta \geq s^* - \sigma_\theta \Phi^{-1}\left(k\left(s^* - F\Phi\left(-z\right)\right)\right) \middle| z\right) \cdot \phi\left(z\right) dz \\
&= \left(D\left(s^*\right) - 1\right) \int_{z=-\infty}^{+\infty} \Phi\left(- \frac{\left\{ \begin{array}{c} \left[s^* - \sigma_\theta \Phi^{-1}\left(k\left(s^* - F\Phi\left(-z\right)\right)\right)\right] \\ - \left(\frac{\tau_{\mu_\theta}}{\tau_{\mu_\theta} + \tau_\theta} \bar{\mu}_\theta + \frac{\tau_\theta}{\tau_{\mu_\theta} + \tau_\theta} s^*\right) \end{array} \right\}}{\sqrt{\frac{1}{\tau_{\mu_\theta} + \tau_\theta}}} \right) \cdot \phi\left(z\right) dz,
\end{aligned}$$

where the second equality follows because $\mu_\theta | z \sim N\left(\frac{\tau_{\mu_\theta}}{\tau_{\mu_\theta} + \tau_\theta} \bar{\mu}_\theta + \frac{\tau_\theta}{\tau_{\mu_\theta} + \tau_\theta} s^*, \frac{1}{\tau_{\mu_\theta} + \tau_\theta}\right)$ under $\sigma_s \to 0$. Similarly, we can work out the RHS of (A.9). Overall, equation (29) is obtained. It is easy to verify the results for the extreme cases of $\sigma_{\mu_\theta} \to 0$ and $\sigma_{\mu_\theta} \to +\infty$ in Lemma 6.

Proof of Proposition 4: Write the function on the LHS of (29) as $V(s^*; \tau_{\mu_\theta})$. The monotonicity of $V(s^*; \tau_{\mu_\theta})$ for the case $\sigma_{\mu_\theta} \to +\infty$ remains for the case of high enough σ_{μ_θ}, and the monotonicity of $V(s^*; \tau_{\mu_\theta})$ for the case $\sigma_{\mu_\theta} \to 0$ remains for the case of low enough $\sigma_{\mu_\theta} > 0$.

Proof of Corollary 1: Write the LHS of (29) as function $V(s^*, \sigma_{\mu_\theta})$. Consider the case of parameter values under which there is a unique equilibrium (e.g., k is high enough; see Proposition 1). The unique equilibrium is stable. Based on the proof of Lemma 2, a stable equilibrium corresponds to $\frac{\partial V(s^*, \sigma_{\mu_\theta})}{\partial s^*} > 0$ at the equilibrium solution s^* to $V(s^*, \sigma_{\mu_\theta}) = 1$.

Then consider the extreme $\sigma_{\mu_\theta} \to 0$, so we have the equilibrium in Proposition 1 (only with μ_θ replaced by $\bar{\mu}_\theta$). In this case, clearly, when $\bar{\mu}_\theta$ is high enough, the unique equilibrium s^* satisfies $s^* < \bar{\mu}_\theta$ (see, e.g., the numerical exercise in Figure 5). For such a high enough $\bar{\mu}_\theta$, we slightly increase σ_{μ_θ} above 0 and have

$$\begin{aligned}
\frac{\partial V(s^*, \sigma_{\mu_\theta})}{\partial \sigma_{\mu_\theta}} = & \left(\int_{-\infty}^{+\infty} \left[\phi\left(\frac{\varkappa\left(s^*, \sigma_{\mu_\theta}, z\right)}{\sqrt{\frac{\left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2}{1 + \left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2}}} \right) \frac{\left(\frac{2\sigma_{\mu_\theta}/\sigma_\theta^2}{\left[1 + \left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2\right]^2} \left(\frac{s^* - \bar{\mu}_\theta}{\sigma_\theta}\right) \right)}{\sqrt{\frac{\left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2}{1 + \left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2}}} \right] \phi\left(z\right) dz \right) \frac{D(s^*) - 1 + \Delta}{\Delta} \\
& + \left(\int_{-\infty}^{+\infty} \left[\phi\left(\frac{\varkappa\left(s^*, \sigma_{\mu_\theta}, z\right)}{\sqrt{\frac{\left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2}{1 + \left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2}}} \right) \left(-\frac{1}{2}\right) \left(\frac{\varkappa\left(s^*, \sigma_{\mu_\theta}, z\right)}{\left(\frac{\left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2}{1 + \left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2}\right)^{\frac{3}{2}}} \right) \frac{\partial \left(\frac{\left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2}{1 + \left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2} \right)}{\partial \sigma_{\mu_\theta}} \right] \phi\left(z\right) dz \right) \frac{D(s^*) - 1 + \Delta}{\Delta},
\end{aligned}$$

where $\varkappa\left(s^*, \sigma_{\mu_\theta}, z\right) \equiv \Phi^{-1}\left(k\left(s^* - F\Phi\left(-z\right)\right)\right) - \frac{1}{1 + \left(\sigma_{\mu_\theta}/\sigma_\theta\right)^2} \left(\frac{s^* - \bar{\mu}_\theta}{\sigma_\theta}\right)$. The first term is negative and the second term approaches zero when $\sigma_{\mu_\theta} \to 0$.[30] Hence, under the sufficient condition that σ_{μ_θ} is close enough to 0, $\frac{\partial V(s^*, \sigma_{\mu_\theta})}{\partial \sigma_{\mu_\theta}} < 0$. By the implicit function theorem, $\frac{\partial s^*}{\partial \sigma_{\mu_\theta}} = -\frac{\partial V(s^*, \sigma_{\mu_\theta})}{\partial \sigma_{\mu_\theta}} / \frac{\partial V(s^*, \sigma_{\mu_\theta})}{\partial s^*} > 0$.

The following is to provide further insight on how aggregate uncertainty affects the amplification mechanism. Denote by s_i^* the threshold used by creditors of bank i and by s^* the threshold used by

[30] The second term under $\sigma_{\mu_\theta} \to 0$ can be simplified as $\lim_{\sigma \to 0} \int_{-\infty}^{+\infty} \phi\left(\frac{z}{\sigma}\right) z \phi\left(z + a\right) dz = 0$.

creditors of all other banks. Then, the equilibrium in Lemma 6 is characterized by the fixed-point problem between s_i^* and s^*, given by the two equations:

$$\left[\int_{z=-\infty}^{+\infty} \Phi\left(-\frac{\left\{\begin{array}{c}\left[s^*-\sigma_\theta \Phi^{-1}\left(k\left(s_i^*-F\Phi(-z)\right)\right)\right] \\ -\left(\frac{\tau_{\mu_\theta}}{\tau_{\mu_\theta}+\tau_\theta}\bar{\mu}_\theta+\frac{\tau_\theta}{\tau_{\mu_\theta}+\tau_\theta}s_i^*\right)\end{array}\right\}}{\sqrt{\frac{1}{\tau_{\mu_\theta}+\tau_\theta}}}\right)\phi(z)\,dz\right]\frac{D(s_i^*)-1+\Delta}{\Delta}=1 \qquad \text{(A.10)}$$

and $s_i^* = s^*$. In fact, to obtain (A.10), the proof of Lemma 6 needs to be revised slightly. That is, (25) is changed to $\frac{\theta^*-\varphi/k}{F} = \Phi\left(\frac{s_i^*-\theta^*}{\sigma_s}\right)$ with $\lim_{\sigma_s\to 0}\varphi(s^*,\mu_\theta) = \Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$, which defines $\theta^* = \theta^*\left(s_i^*;\varphi(s^*,\mu_\theta)\right)$. Then, $\lim_{\sigma_s\to 0} z^* = -\Phi^{-1}\left(\frac{s_i^*-\varphi/k}{F}\right)$ with $\varphi = \Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$ and $\lim_{\sigma_s\to 0}\Pr\left(z \ \geq z^* \middle| z\right) = \Pr\left(z \ \geq -\Phi^{-1}\left(\frac{s_i^*-\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)/k}{F}\right)\middle| z\right) = \Pr\left(\mu_\theta \geq s^*-\sigma_\theta\Phi^{-1}\left(k\left(s_i^*-F\Phi(-z)\right)\right)\middle| z\right)$, where $\mu_\theta|z \sim N\left(\frac{\tau_{\mu_\theta}}{\tau_{\mu_\theta}+\tau_\theta}\bar{\mu}_\theta+\frac{\tau_\theta}{\tau_{\mu_\theta}+\tau_\theta}s_i^*, \frac{1}{\tau_{\mu_\theta}+\tau_\theta}\right)$ under $\sigma_s \to 0$. So (A.10) is obtained.

Write the LHS of (A.10) as function $V(s^*, s_i^*, \sigma_{\mu_\theta})$. It is easy to calculate and verify that $\frac{\partial V}{\partial s^*} < 0$ and $\frac{\partial V}{\partial s_i^*} > 0$, and also $\frac{\partial V}{\partial \tau_{\mu_\theta}} > 0$ when $s_i^* < \bar{\mu}_\theta$ under the sufficient condition that σ_{μ_θ} is close enough to 0. The solution with respect to s_i^* to equation $V(s^*, s_i^*, \sigma_{\mu_\theta}) = 1$ gives the (best response) function $s_i^* = r\left(s^*;\sigma_{\mu_\theta}\right)$. By the implicit function theorem, $\frac{\partial s_i^*}{\partial s^*} > 0$ and $\frac{\partial s_i^*}{\partial \sigma_{\mu_\theta}} > 0$. Also, the slope $\frac{\partial s_i^*}{\partial s^*}$ is lower for a higher σ_{μ_θ} at some relevant points (candidate equilibria) $(s^*, s_i^* = s^*)$.

Proof of Corollary 2: We make an additional assumption: n is decreasing in φ. For simplicity, assume $n = \varphi^{-\beta}$ with $\beta > 0$. In this case, based on Lemma 1, $k = \varphi^{-\beta}\cdot k_0$, where $k_0 \equiv 1/\left(\gamma\sigma_e^2\right)$. Based on the proof of Lemma 6, the equilibrium equation (29) is revised as

$$\left[\int_{z=-\infty}^{+\infty} \Phi\left(\frac{\left\{\begin{array}{c}\left(\frac{\tau_{\mu_\theta}}{\tau_{\mu_\theta}+\tau_\theta}\bar{\mu}_\theta+\frac{\tau_\theta}{\tau_{\mu_\theta}+\tau_\theta}s^*\right) \\ -\left[s^*-\sigma_\theta\Phi^{-1}\left(\left[k_0\left(s^*-F\Phi(-z)\right)\right]^{\frac{1}{1+\beta}}\right)\right]\end{array}\right\}}{\sqrt{\frac{1}{\tau_{\mu_\theta}+\tau_\theta}}}\right)\phi(z)\,dz\right]\frac{D(s^*)-1+\Delta}{\Delta}=1.$$

Write the first term on the LHS as $\Psi\left(z,\sigma_{\mu_\theta}\right) \equiv \int_{z=-\infty}^{+\infty}\Phi\left(\frac{\pi\left(z,\sigma_{\mu_\theta}\right)}{\sqrt{\frac{1}{\tau_{\mu_\theta}+\tau_\theta}}}\right)\phi(z)\,dz$, where $\pi\left(z,\sigma_{\mu_\theta}\right) \equiv \left(\frac{\tau_{\mu_\theta}}{\tau_{\mu_\theta}+\tau_\theta}\bar{\mu}_\theta+\frac{\tau_\theta}{\tau_{\mu_\theta}+\tau_\theta}s^*\right) - \left[s^*-\sigma_\theta\Phi^{-1}\left(\left[k_0\left(s^*-F\Phi(-z)\right)\right]^{\frac{1}{1+\beta}}\right)\right]$. It follows that $\frac{\partial\Psi\left(z,\sigma_{\mu_\theta}\right)}{\partial\sigma_{\mu_\theta}} =$

$$\int_{-\infty}^{+\infty}\left[\phi\left(\frac{\pi\left(z,\sigma_{\mu_\theta}\right)}{\sqrt{\frac{1}{\tau_{\mu_\theta}+\tau_\theta}}}\right)\frac{\frac{\partial\pi\left(z,\sigma_{\mu_\theta}\right)}{\partial\sigma_{\mu_\theta}}}{\sqrt{\frac{1}{\tau_{\mu_\theta}+\tau_\theta}}}\right]\phi(z)\,dz \;+\; \int_{z=-\infty}^{+\infty}\phi\left(\frac{\pi\left(z,\sigma_{\mu_\theta}\right)}{\sqrt{\frac{1}{\tau_{\mu_\theta}+\tau_\theta}}}\right)(-1)\,\frac{\pi\left(z,\sigma_{\mu_\theta}\right)}{\frac{1}{\tau_{\mu_\theta}+\tau_\theta}}\frac{\partial\sqrt{\frac{1}{\tau_{\mu_\theta}+\tau_\theta}}}{\partial\sigma_{\mu_\theta}}\phi(z)\,dz.$$

The first term is the mean channel and the second term is the variance channel. Under the sufficient condition that $\bar{\mu}_\theta - s^*$ is close to 0, the first term is small. We focus on the second term and show that the second term is more likely to be negative when β is higher. First, we show that $\pi\left(z,\sigma_{\mu_\theta}\right)$ is concave in z in a larger range of z when β is higher. In fact,

$$\frac{\partial\pi(z,\sigma_{\mu_\theta})}{\partial z}=\sigma_\theta\Phi^{-1\prime}\left(\left[k_0\left(s^*-F\Phi\left(-z\right)\right)\right]^{\frac{1}{1+\beta}}\right)\frac{1}{1+\beta}\left[k_0\left(s^*-F\Phi\left(-z\right)\right)\right]^{\frac{1}{1+\beta}-1}\left(k_0F\phi\left(-z\right)\right)>0$$

$$\frac{\partial\pi^2\left(z,\sigma_{\mu_\theta}\right)}{\partial z^2}=\sigma_\theta\Phi^{-1\prime\prime}\left(\left[k_0\left(s^*-F\Phi\left(-z\right)\right)\right]^{\frac{1}{1+\beta}}\right)\left[\frac{1}{1+\beta}\left[k_0\left(s^*-F\Phi\left(-z\right)\right)\right]^{\frac{1}{1+\beta}-1}\left(k_0F\phi\left(-z\right)\right)\right]^2$$
$$+\sigma_\theta\Phi^{-1\prime}\left(\left[k_0\left(s^*-F\Phi\left(-z\right)\right)\right]^{\frac{1}{1+\beta}}\right)\left(\begin{array}{c}\frac{1}{1+\beta}\left(\frac{1}{1+\beta}-1\right)\left[k_0\left(s^*-F\Phi\left(-z\right)\right)\right]^{\frac{1}{1+\beta}-2}\cdot\left(k_0F\phi\left(-z\right)\right)^2\\+\frac{1}{1+\beta}\left[k_0\left(s^*-F\Phi\left(-z\right)\right)\right]^{\frac{1}{1+\beta}-1}\cdot\left(-k_0F\phi'\left(-z\right)\right)\end{array}\right),$$

where $\frac{1}{1+\beta}\left(\frac{1}{1+\beta}-1\right)<0$. Second, consider the general case $\int_{z=-\infty}^{+\infty}\phi\left(\frac{\pi(z)}{\sigma}\right)\pi\left(z\right)dz$, where $\pi\left(z\right)$ is increasing and concave, with the range of π being $(-\infty,+\infty)$. It follows that $\int_{z=-\infty}^{+\infty}\phi\left(\frac{\pi(z)}{\sigma}\right)\pi\left(z\right)dz=\int_{y=-\infty}^{+\infty}\phi\left(\frac{y}{\sigma}\right)yd\pi^{-1}\left(y\right)>0$ under the sufficient condition that σ is low enough, by considering $d\pi^{-1}\left(y\right)/dy>0$ and $d^2\pi^{-1}\left(y\right)/dy^2>0$. Third, when $\frac{1}{\tau_{\mu_\theta}+\tau_\theta}$ is small enough, the sign of the second term in $\frac{\partial\Psi(z,\sigma_{\mu_\theta})}{\partial\sigma_{\mu_\theta}}$ is determined by the sign of $\int_{z=-\infty}^{+\infty}\phi\left(\frac{\pi(z,\sigma_{\mu_\theta})}{\sqrt{\frac{1}{\tau_{\mu_\theta}+\tau_\theta}}}\right)(-1)\frac{\pi(z,\sigma_{\mu_\theta})}{\frac{1}{\tau_{\mu_\theta}+\tau_\theta}}\frac{\partial\sqrt{\frac{1}{\tau_{\mu_\theta}+\tau_\theta}}}{\partial\sigma_{\mu_\theta}}dz$, so the sign is more likely to be negative when β is higher.

Proof of Corollary 3: Based on equation (29), for a given σ_{μ_θ}, when σ_θ is small enough or large enough, there exists a unique equilibrium; when σ_θ is in an intermediate range, multiple equilibria can exist. Hence, fixing σ_{μ_θ} and letting σ_θ start from $\sigma_\theta=0^+$ and increase, there exists a critical value $\sigma_\theta=\sigma_\theta^*$ below which the equilibrium is unique. So around $\left(\sigma_{\mu_\theta},\sigma_\theta^*\right)$, we can find the case such that when both σ_{μ_θ} and σ_θ increase slightly and there is a tiny decrease for $\sigma_{\mu_\theta}/\sigma_\theta$, the equilibrium switches from uniqueness to multiplicity.

Proof of Lemmas 7 and 8: The proof is straightforward and hence omitted.

Proof of Proposition 5: The expected utility for the outside investor sector is given by $n\mathbb{E}\left[U(W^j)|\left\{l_i\right\}\right]=-n\exp\left(-\gamma\int q_i(\theta_i-l_i)di+\frac{1}{2}\gamma^2\sigma_e^2\left(\int q_idi\right)^2\right)=-n\exp\left(-\gamma\left(\left(\alpha\eta\varphi\right)/k\right)\cdot\left(\frac{\alpha\eta\varphi}{n}\right)+\frac{1}{2}\gamma^2\sigma_e^2\left(\frac{\alpha\eta\varphi}{n}\right)^2\right)=-n\exp\left(-\frac{1}{2}\left(\gamma/k\right)\frac{1}{n}\left(\alpha\eta\varphi\right)^2\right)$, by considering $\theta_i-l_i=\left(\alpha\eta\varphi\right)/k$, $n\int q_idi=\alpha\eta\varphi$ and $k\equiv n/\left(\gamma\sigma_e^2\right)$. Also, $\mathbb{E}\left(l_i\left(\theta_i\right)|\theta_i\in\left(s^{**},s^*\right)\right)>l_i\left(\theta_i=s^{**}\right)=\frac{s^{**}}{I}>\mathbb{E}\left(\theta_i\,|\theta_i\leq s^{**}\right)/I$. And $Y\left(Q_1,Q_2\right)=\left(\mu_\theta+c\right)-\left(\alpha\eta\varphi\right)^2/k$. Due to the complexity of the general-equilibrium nature of (35) with many endogenous variables, we show the existence of cases i) to iii) under the same set of parameter values. The simulation exercise confirms the existence for a large parameter space. The tradeoff between Q_1 and Q_2 in reducing inefficient fire sales is discussed after Figure 11.

Function $Y\left(Q_1,Q_2\right)$ is continuous in $Q_1\in(0,Q)$ and in $Q_2\in(0,Q)$ for a low Q. Hence, the optimization problem (35) has solutions in a closed set $Q_1\in\left[\delta,Q-\delta\right]$, where δ is an arbitrarily small positive number. Note that $\frac{\partial Y}{\partial Q_1}$ is decreasing in Q_1 and $\frac{\partial Y}{\partial Q_2}$ is relatively less sensitive to Q_2. When $\left(\mu_\theta,k\right)$ is in the region such that $\varphi=\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$ is high enough, $\left.\frac{\partial Y/\partial Q_1}{\partial Y/\partial Q_2}\right|_{(Q_1,Q_2)=(Q_1,Q-Q_1)}>1$ at any $Q_1\in[0,Q]$, so the optimal allocation is $\left(Q_1,Q_2\right)=\left(Q,0\right)$. When $\left(\mu_\theta,k\right)$ is in the region such that $\varphi=\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$ is low enough, $\left.\frac{\partial Y/\partial Q_1}{\partial Y/\partial Q_2}\right|_{(Q_1,Q_2)=(Q_1,Q-Q_1)}<1$ at any $Q_1\in[0,Q]$, so the optimal allocation is $\left(Q_1,Q_2\right)=\left(0,Q\right)$. When $\left(\mu_\theta,k\right)$ is in the region such that $\varphi=\Phi\left(\frac{s^*-\mu_\theta}{\sigma_\theta}\right)$ is in an intermediate range, there exists $Q_1\in(0,Q)$ such that $\left.\frac{\partial Y/\partial Q_1}{\partial Y/\partial Q_2}\right|_{(Q_1,Q_2)=(Q_1,Q-Q_1)}=1$ and the optimal Q_1 is typically unique.

A minor complication is that when (μ_θ, k) is in the region such that $\varphi = \Phi\left(\frac{s^* - \mu_\theta}{\sigma_\theta}\right)$ is low enough (case ii), function $Y(Q_1, Q_2)$ exhibits a discontinuity at $Q_2 = 0$ due to the normal distribution of $\theta_i \sim N(\mu_\theta, \sigma_\theta^2)$; in fact, based on (15), when $c = 0$, $s^{**} = -\infty$; when $c = 0^+$, $s^{**} > 0$. However, because around this discontinuity $Y(Q_1, Q_2 = 0) < Y(Q_1, Q_2 = 0^+)$ is true, the optimum $Q_2 = Q$ in case ii) will not change after taking into account this local discontinuity.

Proof of Corollary 4: Based on Propositions 3 and 5, the conclusion is easy to obtain.

B Additional Results

B.1 Equilibrium at $t = 0$ with Endogenous Liquid Asset Holdings

In this subsection, we endogenize liquid asset holdings, study the optimal level of liquidity holdings for banks, and address the question of whether individual banks' choice is socially optimal. Our paper is the first to study this question in general equilibrium in a global-games framework.[31]

We study the ex ante problem at $t = 0$ of the baseline model. Specifically, no longer assume that the asset side of the balance sheet of a bank is exogenously given. Rather, a bank makes a portfolio choice $(c, 1-c)$ at $t = 0$, where c denotes the amount of liquid asset holdings ("cash") and $1 - c$ denotes the units of risky assets. The liability side is still given by $(F, 1 - F)$.

With the above setup, our model endogenizes c (as well as R). We study the decentralized equilibrium for individual banks and the social planner's constrained problem, in order.[32]

Decentralized Competitive Equilibrium. The decentralized competitive equilibrium at $t = 0$ consists of the following two elements:

(i) Taking the fire-sale price function $\hat{l}(\cdot)$ and the interbank return $\tilde{I}$ at $t = 1$ as given, an individual bank i chooses its optimal c_i, subject to its creditors' participation.

(ii) Given that all banks choose the same c in symmetric equilibrium (i.e., $c_i = c$ for $i \in [0, 1]$), the market equilibrium determines $\hat{l}(\cdot)$ and $\tilde{I}$.

In other words, the competitive equilibrium is characterized by a fixed-point problem between c and $\left(\hat{l}(\cdot), \tilde{I}\right)$. We proceed to find the competitive equilibrium in two steps, corresponding to elements (i) and (ii). To reduce notational clutter and without causing confusion, we drop subscript i for the bank index in notations unless otherwise specified.

[31] Early works not using the global-games approach either assume that banks face exogenous idiosyncratic liquidity shocks and do not model the run probability (e.g., Bhattacharya and Gale (1987), Allen and Gale (2000)) or they assume corporate liquidity shocks instead of studying run risk (e.g., Kara and Ozsoy (2020)). A few recent works that use the global-games approach are partial-equilibrium models; for example, Vives (2014a) and Carletti, Goldstein, and Leonello (2020) treat all banks in the system as having the same fundamentals and Ahnert (2016) studies two banks without considering the welfare of the asset buyer sector and assumes an exogenous debt contract. Liu (2016) studies heterogeneous banks but focuses only on the planner's choice.

[32] To examine the ex ante problem, for simplicity we can choose parameter values (i.e., some (μ_θ, k)) that guarantee a unique equilibrium ex post at $t = 1$ (see Proposition 1). In this case, emergence of multiple equilibria ex post can be interpreted as the outcome of an "unexpected" shock (to μ_θ or to k) as in Kiyotaki and Moore (1997) and Allen and Gale (2000). Alternatively, following the literature (e.g., Jeanne and Korinek (2018)), in the case of the existence of multiple equilibria ex post, agents have a view ex ante on which equilibrium will be selected (e.g., the least-efficient equilibrium under the most pessimistic views with Knightian uncertainty in the spirit of Caballero and Krishnamurthy (2008) or the most-efficient equilibrium by anticipating the intervention of the government).

Step 1: Find an individual bank's optimal choice of c for a given $\left(\hat{l}(\cdot), \tilde{I}\right)$.

If an individual bank i chooses $(c, 1-c)$, what are its creditors' rollover strategy and participation condition? Given that the creditors anticipate $\hat{l}(\cdot)$ and $\tilde{I}$, the rollover threshold s^* is determined by (18). The participation condition (IR) for a creditor is given by

$$\underbrace{\int_{\theta_i=-\infty}^{s^*} \frac{(1-c)\,\hat{l}_i + c}{F} \cdot d\Phi\left(\frac{\theta_i - \mu_\theta}{\sigma_\theta}\right)}_{\text{in the case of bank failing at } t=1} + \underbrace{\int_{\theta_i=s^*}^{+\infty} \hat{D}(\theta_i; R) \cdot d\Phi\left(\frac{\theta_i - \mu_\theta}{\sigma_\theta}\right)}_{\text{in the case of bank surviving to } t=2} = R_0, \tag{B.1}$$

where the exogenous R_0 is the reserve return of lending for bank creditors. In (B.1), recalling Figure 3, two of the payoff cases are on the equilibrium path, while the other two are off the equilibrium path. Equations (18) and (B.1) jointly give the mapping $\left(c, \left(\hat{l}(\cdot), \tilde{I}\right)\right) \rightarrow (s^*, R)$.

An individual bank's optimal choice of c is therefore given by the program

$$\begin{aligned} c^* \;&=\; \arg\max_c \underbrace{\int_{\theta_i=-\infty}^{s^*} \left[(1-c)\,\hat{l}_i + c\right] \cdot d\Phi\left(\frac{\theta_i - \mu_\theta}{\sigma_\theta}\right)}_{\text{in the case of bank failing at } t=1} + \underbrace{\int_{\theta_i=s^*}^{+\infty} \left[(1-c)\,\theta_i + cI\right] \cdot d\Phi\left(\frac{\theta_i - \mu_\theta}{\sigma_\theta}\right)}_{\text{in the case of bank surviving to } t=2} \\ &\quad \text{s.t. (16), (17), (18), and (B.1).} \end{aligned} \tag{B.2}$$

The objective function in (B.2) is to maximize the bank's equity value. This is equivalent to maximizing the total value of the bank (i.e., its debt value plus equity value), because the creditors of a bank, in total, claim a constant residual value, FR_0.[33] As for the constraints, the bankowner anticipates the liquidation price rule (16) and the interbank return (17) and takes into account creditors' response (18) and (B.1).

Denote by $Y(c)$ the objective function in program (B.2). The first-order condition is

$$\frac{dY}{dc} = \frac{\partial Y}{\partial s^*}\frac{\partial s^*}{\partial c} + \frac{\partial Y}{\partial c} = 0. \tag{B.3}$$

The tradeoff is as follows. Holding more cash reduces the probability of a run ex post and thus the chance of forced fire sales, which is the gain. On the other hand, holding more cash wastes the valuable investment opportunity as the risky asset has a higher expected return, which is the loss.

Step 2: Find the equilibrium $\left(\hat{l}(\cdot), \tilde{I}\right)$ for a given c chosen by all other banks.

Suppose all other banks choose c. The market equilibrium then determines $(s^*, (s^{**}, \{l_i\}, I), \varphi, R)$, which in turn determines $\left(\hat{l}(\cdot), \tilde{I}\right)$, that is,

$$\left.\begin{array}{ll} \text{(13) to (15)} & \text{(asset prices)} \\ \text{(18)} & \text{(creditor run)} \\ \text{(B.1)} & \text{(IR of creditors))} \\ \text{(12)} & \text{(aggregate liquidation)} \end{array}\right\}. \tag{B.4}$$

[33] To focus solely on the friction we are interested in, we assume that a bank's choice $(c, 1-c)$ is observable by or contractible with its creditors. This means that R is contingent on c, so the bankowner fully internalizes the impact of his choice of c on the debt value and thus maximizes the total value of the bank.

Note that (B.4) is essentially Proposition 2 together with the participation condition (B.1).

By symmetric equilibrium across banks, we have

$$c^* = c. \tag{B.5}$$

Constrained Second-Best Equilibrium. In the constrained second-best equilibrium, the social planner chooses $(c, 1-c)$ on behalf of individual banks. She takes the fire-sale price function $\hat{l}(\cdot)$ and the interbank return $\tilde{I}$ as *endogenous* and recognizes that the choice of $(c, 1-c)$ endogenously impacts $\left(\hat{l}(\cdot), \tilde{I}\right)$. Her optimal choice of c is given by

$$c^{*SB} = \arg\max_c \left[\begin{array}{l} \int_{\theta_i=-\infty}^{s^*} \left[(1-c)\,\hat{l}_i + c\right] \cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right) + \int_{\theta_i=s^*}^{+\infty} \left[(1-c)\,\theta_i + cI\right] \cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right) \\ +\mathbb{E}\left[U(W^j)|\,\{l_i\}\right] \end{array} \right]$$

$$\text{s.t. (13) to (15), (18), (B.1), and (12),} \tag{B.6}$$

where $\mathbb{E}\left[U(W^j)|\,\{l_i\}\right] = -\exp\left(-\frac{1}{2}\gamma\,(\eta\varphi)^2/k\right) \equiv Y_o$ is the expected utility for the outside investor sector.[34] In program (B.6), the objective function is to maximize the aggregate value of the entire banking sector and the outside investor sector in general equilibrium.[35] The former includes the first term (the failing banks at $t=1$) and the second term (the surviving banks at $t=2$), and the latter corresponds to the third term in the objective function. Program (B.6) contains constraints (13) to (15) as well as (12), which endogenously determine the asset prices, while program (B.2) does not. The constraints of program (B.6) give the mapping $c \to (s^*, (s^{**}, \{l_i\}, I), \varphi, R)$.

Denote by $Y_s(c)$ the objective function in program (B.6). Clearly, $Y_s = Y + Y_o$, where Y, defined before (B.3), is the objective function in program (B.2). The first-order condition is

$$\frac{dY_s}{dc} = \frac{\partial Y}{\partial s^*}\frac{ds^*}{dc} + \frac{\partial Y}{\partial c} + \left(\frac{\partial Y}{\partial(\eta\varphi)}\frac{d(\eta\varphi)}{dc} + \frac{\partial Y}{\partial I}\frac{dI}{dc}\right) + \frac{\partial Y_o}{\partial(\eta\varphi)}\frac{d(\eta\varphi)}{dc} = 0, \tag{B.7}$$

where the first two terms have counterparts in (B.3), the third and fourth terms are the indirect effects through endogenous prices $\left(\hat{l}(\cdot), \tilde{I}\right)$, and the last term is the effect on the expected utility of the outside investor sector.

We see a key difference between the two first-order conditions (B.3) and (B.7). Namely, $\frac{\partial s^*}{\partial c}$ in (B.3) is determined by the constraints of program (B.2), while $\frac{ds^*}{dc}$ in (B.7) is determined by those of program (B.6). To grasp some intuition, let us look at one constraint — equation (18) for the creditor-run equilibrium, rewritten as $\frac{c+(1-c)\cdot l(s^*)}{F} \cdot \frac{\hat{D}(s^*)-1+\Delta}{\Delta} = 1$, where $l(s^*) = s^* - (\eta\varphi)/k$. For individual banks, s^* is a function of c while $(\eta\varphi)/k$ is exogenous, which gives $\frac{\partial s^*}{\partial c}$. In contrast, for the social planner, discount $(\eta\varphi)/k$ is endogenous and is a function of c and hence a change in c has an additional indirect effect on s^* through $\eta\varphi$, and the total effect corresponds to $\frac{ds^*}{dc}$.

Under certain conditions, $c^* < c^{*SB}$. That is, individual banks' optimal level of liquid asset holdings in the decentralized equilibrium is lower than the constrained social optimum.

Intuitively, individual banks do not internalize that a higher level of liquidity holdings of their own has a positive effect on other banks — it reduces the run probability of their own and thus low-

[34] For simplicity and without loss of generality, we normalize $n = 1$ (the mass of outside investors) in this subsection.

[35] Mas-Colell, Whinston, and Green (1995) give a definition for a welfare function of an economy in which different types of agents can have different preference functions. Weights for different types of agents can be different.

ers the pressure on the fire-sale price discount in the asset market, thereby making other banks less likely to suffer runs and consequently reducing inefficient fire sales in the system. More concretely, when asset prices enter the constraints of the optimization problem of private agents, pecuniary externality that operates through asset prices can arise (e.g., Dávila and Korinek (2018), Brunnermeier, Eisenbach, and Sannikov (2013)). Specifically, in our model, a wedge exists between the two first-order conditions (B.3) and (B.7), due to two sources of externality. First, as discussed earlier, $\frac{\partial s^*}{\partial c}$ in (B.3) is different from $\frac{ds^*}{dc}$in (B.7), corresponding to "collateral externality" in Dávila and Korinek (2018). Second, sellers (banks) are risk-neutral while buyers (outside investors) are risk-averse in our model. The difference in MRS causes "distributive externality" as in Dávila and Korinek (2018). Overall, under the sufficient condition that the effect of the first externality dominates, it follows that $\frac{dY_s}{dc}\big|_{c=c^*} > \frac{\partial Y}{\partial c}\big|_{c=c^*} = 0$, which implies $c^* < c^{*SB}$.

Finally, our model can also address two other questions. First, if the government cannot commit to avoiding ex post interventions in the asset market, what happens? In this case, individual banks would hold less liquidity ex ante in the decentralized equilibrium. In fact, equation (B.4) will be adjusted as in (34). The government's liquidity support ex post will absorb a fraction of sold assets in the secondary asset market. The liquidation prices will hence be raised and the run threshold s^* will decrease, and consequently individual banks have incentives to hold less liquidity ex ante. Second, it is easy to extend the study in this section to the case with aggregate uncertainty. In fact, this involves two changes. The participation condition in (B.1) becomes

$$\int_{\mu_\theta=-\infty}^{+\infty} \underbrace{\int_{\theta_i=-\infty}^{s^*} \frac{(1-c)\,\hat{l}_i\,(\mu_\theta)+c}{F} \cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right)}_{\text{in the case of bank failing at } t=1} + \underbrace{\int_{\theta_i=s^*}^{+\infty} \hat{D}(\theta_i;R)\cdot d\Phi\left(\frac{\theta_i-\mu_\theta}{\sigma_\theta}\right)}_{\text{in the case of bank surviving to } t=2} d\Phi\left(\frac{\mu_\theta-\bar{\mu}_\theta}{\sigma_{\mu_\theta}}\right) = R_0$$

and the ex post creditor run-asset market equilibrium at $t = 1$ becomes (29).

B.2 Alternative Creditor-Run Payoff Structure

First, we show that our model results are robust under the full payoff structure in Figure B1 below.

	Total calling proportion $\lambda \in [0, \frac{l_i}{F})$ (bank survives)	Total calling proportion $\lambda \in [\frac{l_i}{F}, 1]$ (bank fails)
Hold	$\min\left[R, \frac{\left(1-\frac{F\lambda}{l_i}\right)}{(1-\lambda)F} v_i\right]$	$\frac{l_i}{F} - \Delta$
Call	1	$\frac{l_i}{F}$

Figure B1. Full creditor-run payoff structure.

If $\lambda \in [0, \frac{l_i}{F})$, the bank needs to liquidate $\frac{F\lambda}{l_i}$ units of its assets to raise cash to pay its $F\lambda$ calling creditors. Thus, at $t = 2$, $1 - \frac{F\lambda}{l_i}$ units of assets remain. Since the number of staying creditors at $t = 2$ is $(1-\lambda)\,F$, these creditors' total notional claim is $(1-\lambda)\,FR$. Hence, a staying creditor will have a realized payoff $\frac{\min\left[(1-\lambda)FR,\left(1-\frac{F\lambda}{l_i}\right)v_i\right]}{(1-\lambda)F} = \min\left[R, \frac{1-\frac{F\lambda}{l_i}}{(1-\lambda)F} v_i\right]$ at $t = 2$, as in Diamond and Dybvig (1983). Notice that by setting $\lambda = 0$, the payoff $\min\left[R, \frac{1-\frac{F\lambda}{l_i}}{(1-\lambda)F} v_i\right]$ becomes $\min\left[R, \frac{v_i}{F}\right]$, so

Figure B1 becomes exactly the same as Figure 3.

Under the payoff structure in Figure B1, we prove that (7) is replaced by

$$\int_0^{\frac{s^*-\varphi/k}{F}}\left[\mathbb{E}\left(\min\left[R,\frac{\left(1-\frac{F\lambda}{s^*-\varphi/k}\right)}{(1-\lambda)F}v_i\right]|\theta_i=s^*\right)-1\right]d\lambda=\left(1-\frac{s^*-\varphi/k}{F}\right)\Delta, \tag{B.8}$$

where $v_i \sim N(\theta_i, \sigma_e^2)$. Notice that if we set $\lambda = 0$ inside the integral, (B.8) becomes identical to (7). We show that under a sufficient condition the following key property of (7) in Lemma 3 preserves for (B.8): $\frac{\partial s^*}{\partial \varphi} > 0$. Therefore, the results of the model change only quantitatively, not qualitatively.

Proof: Under the alternative payoff structure in Figure B1, (5) is replaced by

$$\begin{aligned}&\int_{\theta_i=\theta^*}^{+\infty}\left(\mathbb{E}\left(\min\left[R,\frac{\left(1-\frac{F\lambda(\theta_i;s^*)}{l_i}\right)}{(1-\lambda(\theta_i;s^*))F}v_i\right]|\theta_i\right)-1\right)d\Phi\left(\frac{\theta_i-\left(\frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta+\frac{\tau_s}{\tau_\theta+\tau_s}s^*\right)}{\sqrt{\frac{1}{\tau_\theta+\tau_s}}}\right)\\&=\int_{\theta_i=-\infty}^{\theta_i=\theta^*}\Delta d\Phi\left(\frac{\theta_i-\left(\frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta+\frac{\tau_s}{\tau_\theta+\tau_s}s^*\right)}{\sqrt{\frac{1}{\tau_\theta+\tau_s}}}\right),\end{aligned} \tag{B.9}$$

where $l_i = \theta_i - \varphi/k$, $\lambda(\theta_i; s^*) = \Phi\left(\frac{s^*-\theta_i}{\sigma_s}\right)$, $v_i \sim N(\theta_i, \sigma_e^2)$ and $\theta_i|s^* \sim N(\frac{\tau_\theta}{\tau_\theta+\tau_s}\mu_\theta + \frac{\tau_s}{\tau_\theta+\tau_s}s^*, \frac{1}{\tau_\theta+\tau_s})$.

Combining (B.9) and (4) into one equation and using the same method as that in the proof of Lemma 2 of changing variables of the integral, we can transform the combined equation for the limiting case of $\sigma_s \to 0$ into (B.8).

Next, we prove that (B.8) retains the property $\frac{\partial s^*}{\partial \varphi} < 0$ of (7). Write (B.8) as $\Lambda(s^*, \varphi) \equiv \int_0^{\frac{s^*-\varphi/k}{F}}\left[\mathbb{E}\left(\min\left[R,\frac{\left(1-\frac{F\lambda}{s^*-\varphi/k}\right)}{(1-\lambda)F}v_i\right]|\theta_i=s^*\right)-1\right]d\lambda-\left(1-\frac{s^*-\varphi/k}{F}\right)\Delta=0$. This implies $\frac{\partial\Lambda}{\partial s^*} = \int_0^{\frac{s^*-\varphi/k}{F}}\left[\left.\frac{\partial\mathbb{E}\left(\min\left[R,\frac{\left(1-\frac{F\lambda}{s^*-\varphi/k}\right)}{(1-\lambda)F}v_i\right]|\theta_i\right)}{\partial\theta_i}\right|_{\theta_i=s^*}\right]d\lambda+\int_0^{\frac{s^*-\varphi/k}{F}}\left[\mathbb{E}\left(\frac{\partial\min\left[R,\frac{\left(1-\frac{F\lambda}{s^*-\varphi/k}\right)}{(1-\lambda)F}v_i\right]}{\partial s^*}|\theta_i=s^*\right)\right]d\lambda+ \left(-\frac{1}{F}\right)+\frac{1}{F}\Delta$, where $\frac{1-\frac{F\lambda}{s^*-\varphi/k}}{(1-\lambda)F} \in \left[0, \frac{1}{F}\right]$ for $\lambda \in \left[0, \frac{s^*-\varphi/k}{F}\right]$. The first term is positive because the distribution of $\frac{\left(1-\frac{F\lambda}{s^*-\varphi/k}\right)}{(1-\lambda)F}v_i$ under a higher θ_i has first order stochastic dominance over that under a lower θ_i. In the second term, $\frac{\partial\frac{\left(1-\frac{F\lambda}{s^*-\varphi/k}\right)}{(1-\lambda)F}}{\partial s^*} > 0$ and, therefore, the second term is positive under a sufficient condition that σ_e is small enough which ensures $\Pr(v_i < 0|\theta_i = s^*)$ is small enough. The sum of the third term and the fourth term approaches 0 when $\Delta \to 1$. Overall, $\frac{\partial\Lambda}{\partial s^*} > 0$ under a sufficient condition that σ_e is small enough and Δ is close enough to 1. Similarly, $\frac{\partial\Lambda}{\partial\varphi} = \int_0^{\frac{s^*-\varphi/k}{F}}\left[\mathbb{E}\left(\frac{\partial\min\left[R,\frac{\left(1-\frac{F\lambda}{s^*-\varphi/k}\right)}{(1-\lambda)F}v_i\right]}{\partial\varphi}|\theta_i=s^*\right)\right]d\lambda+\frac{1/k}{F}+\left(-\frac{1/k}{F}\Delta\right)$ and $\frac{\partial\Lambda}{\partial\varphi} < 0$ under a sufficient condition that σ_e is small enough and Δ is close enough to 1. Hence, by the implicit function theorem, it follows that $\frac{\partial s^*}{\partial\varphi} = -\frac{\partial\Lambda/\partial\varphi}{\partial\Lambda/\partial s^*} > 0$. Q.E.D.

Second, we show that our model is robust under the alternative payoff structure of Rochet and

Vives (2004). As in Rochet and Vives (2004), each creditor of a bank is an institutional investor (a fund), run by its fund manager. A fund manager has the following compensation scheme. If the fund manager calls his fund's investment at $t = 1$, his payoff is a constant w_0, or the face value 1 multiplied by proportion w_0. If, instead, the fund manager holds the investment at $t = 1$, he will obtain compensation w *conditional on* his fund's investment not defaulting (i.e., the investment return is no less than R), where $w > w_0$. Hence, the payoff structure is as given in Figure B2.

	Total calling proportion $\lambda \in [0, \frac{l_i}{F})$ (bank survives)	Total calling proportion $\lambda \in [\frac{l_i}{F}, 1]$ (bank fails)
Hold	$w \cdot \Phi\left(\frac{\left(1-\frac{F}{l_i}\lambda\right)\theta_i-(1-\lambda)FR}{\left(1-\frac{F}{l_i}\lambda\right)\sigma_e}\right)$	0
Call	w_0	w_0

Figure B2. Alternative creditor-run payoff structure.

If $\lambda \in [\frac{l_i}{F}, 1]$, a creditor run occurs and the bank fails at $t = 1$; its staying creditors get nothing and thus their fund manager's compensation is 0 because of the default. If $\lambda \in [0, \frac{l_i}{F})$, the bank must liquidate $\frac{F\lambda}{l_i}$ units of its assets to raise cash to pay its $F\lambda$ calling creditors. Thus, at $t = 2$, $1 - \frac{F\lambda}{l_i}$ units of assets will remain, the payoff distribution of which, conditional on θ_i, is $\left(1 - \frac{F\lambda}{l_i}\right) v_i \sim N\left(\left(1 - \frac{F\lambda}{l_i}\right)\theta_i, \left(1 - \frac{F\lambda}{l_i}\right)^2 \sigma_e^2\right)$. Since the number of staying creditors at $t = 2$ is $(1 - \lambda) F$, these creditors' total notional claim is $(1 - \lambda) FR$. Hence, the probability that the bank will not default to these creditors at $t = 2$ conditional on θ_i is $\Phi\left(\frac{\left(1-\frac{F\lambda}{l_i}\right)\theta_i-(1-\lambda)FR}{\left(1-\frac{F\lambda}{l_i}\right)\sigma_e}\right)$.

Under the payoff structure in Figure B2, (7) is instead replaced by $\int_0^{\frac{s^*-\varphi/k}{F}} \Phi\left(\frac{\left(1-\frac{F}{s^*-\varphi/k}\lambda\right)s^*-(1-\lambda)FR}{\left(1-\frac{F}{s^*-\varphi/k}\lambda\right)\sigma_e}\right) d\lambda = \frac{w_0}{w}$. It is easy to prove that the following key property of (7) in Lemma 3 remains: $\frac{\partial s^*}{\partial \varphi} > 0$. Moreover, Proposition 1 still holds, and thus all the results carry through.

Similar to Figure 3, we can follow Morris and Shin (2009) to simplify the payoff structure of the creditor-run game in Rochet and Vives (2004), which gives Figure B3.

	Total calling proportion $\lambda \in [0, \frac{l_i}{F})$ (bank survives)	Total calling proportion $\lambda \in [\frac{l_i}{F}, 1]$ (bank fails)
Hold	$w \cdot \Phi\left(\frac{\theta_i-FR}{\sigma_e}\right)$	0
Call	w_0	w_0

Figure B3. Simplified alternative creditor-run payoff structure.

Under Figure B3, (7) is replaced by $\frac{l_i(\theta_i=s^*)}{F} \cdot w\Phi\left(\frac{s^*-FR}{\sigma_e}\right) = w_0$, where $l_i(\theta_i = s^*) = s^* - \varphi/k$ by (2). It is easy to show that the property $\frac{\partial s^*}{\partial \varphi} > 0$ is preserved.